**EARTH'S CRYOSPHERE**
SCIENTIFIC JOURNAL

*Earth's Cryosphere*, 2022, vol. XXVI, No. 3, pp. 26–36

*PHYSICAL AND CHEMICAL PROCESSES IN FROZEN GROUND AND ICE*

# ANALYTICAL REVIEW OF APPROACHES AND METHODS OF MATHEMATICAL MODELING OF THE PROCESSES OF SOIL FREEZING AND HEAVING

**E.V. Safronov[1], V.G. Cheverev[1,*], A.V. Brouchkov[1], S.N. Buldovich[1], V.Z. Khilimonyuk[1], L.V. Grigoriev[2], E.M. Makarycheva[2], E.V. Gnilomedov[2]**

[1] *Lomonosov Moscow State University, Faculty of Geology, Department of Geocryology, Leninskie gory 1, Moscow, 119991 Russia*
[2] *Scientific Research Institute of Pipeline Transport (LLC NII Transneft), Department of External Impact Modeling and Stability Calculations of Monitoring Objects, Sevastopolskii prosp. 47a, Moscow, 117186 Russia*

[*] *Corresponding author; e-mail: cheverev44@mail.ru*

The article provides analytical review of existing approaches and specific models for solving problems of freezing, thawing, and frost heaving of soils. The authors analyzed about 100 published works of Russian and 100 works of foreign authors, including articles, monographs, dissertations, patents, conference proceedings, scientific reports. Special attention in the analysis of the physical formulation of mathematical models is paid to taking into account the mechanism of heat and mass transfer, ice segregation, phase transitions of pore water, and the development of deformations and forces of frost heaving of freezing soils.

***Keywords:*** *mathematical modeling, soil freezing, freezing front, frost heaving, cryogenic migration, heat and mass transfer, water phase transitions, shrinkage.*

***Recommended citation:*** Safronov E.V., Cheverev V.G., Brouchkov A.V., Buldovich S.N., Khilimonyuk V.Z., Grigoriev L.V., Makarycheva E.M., Gnilomedov E.V., 2022. Analytical review of approaches and methods of mathematical modeling of the processes of soil freezing and heaving. *Earth's Cryosphere*, XXVI (3), 26–36.

## INTRODUCTION

The process of soil freezing is primarily a process of heat and moisture transfer and pore water phase transition into ice. For the physical formulation of mathematical modeling of the process, it is necessary to solve the thermal problem with phase transitions and the problem of moisture transfer in the soil. To solve the thermal problem, the Fourier heat equation is the basic approach. In the three-dimensional version, it can be written as [*Tikhonov, Samarsky, 1999*]:

$$C\rho\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)+f\left(x,y,z,t\right), \qquad (1)$$

where $\lambda$ is thermal conductivity, W/(m·K); $T$ is temperature, °C; $t$ is time, s; $f(x,y,z,t)$ is the so-called function of heat sources; $C$ is the specific heat capacity, J/kg; $x$, $y$, $z$ are coordinates, m; and $\rho$ is density of the material, kg/m$^3$.

The application of the Fourier equation is valid for a continuous medium. When the soil freezes, there is a boundary between the thawed and frozen states, at which the phase composition of water can change abruptly. In this case, we can talk about the interface between the thawed and frozen soil with different thermophysical characteristics, which imposes an additional boundary condition on Eq. (1).

To solve the thermal problem with a clear boundary between the liquid and solid phases in the presence of the "jump" of moisture at the boundary, the classical formulation of the Stefan problem is used. One of the variants of writing the heat balance equation directly at the freezing front is given, for example, in [*Brovka, 1991*]*:

$$\left(w_{tot}-w_{w}\right)L\rho_d\frac{d\xi}{dt}=\lambda_U\left.\frac{dT}{dx}\right|_U-\lambda_F\left.\frac{dT}{dx}\right|_F, \qquad (2)$$

where $L$ is the specific heat of ice melting, J/kg; $\rho_d$ is the density of the frozen soil skeleton, kg/m$^3$; $w_{tot}$ is

* In this paper, as well as in the works of Russian and foreign authors, conventional signs for the same physical values do not always coincide. We present them in the original form with necessary explanation.

the total weight moisture content, f.u. (fraction of a unit); $w_w$ is the specific content of unfrozen water in frozen soil at the freezing front, f.u.; $\frac{d\xi}{dt}$ is the change in the thickness of the frozen part of the soil over time, m/s; $\left.\frac{dT}{dx}\right|_U$, $\left.\frac{dT}{dx}\right|_F$ are temperature gradients in thawed and frozen zones, K/m; $\lambda_U$, $\lambda_F$ is the thermal conductivity of the soil in the thawed and frozen zones, W/(m·K).

The solution of the Stefan problem with various boundary conditions is considered in [*Kudryavtsev, 1978*]. Generalized analytical solution of this problem is suggested in [*Getz, Meirmanov, 2000*].

The application of the Fourier equation for the selected soil zones and the equations of thermal and material balance at the boundaries of these zones allows us to describe the processes of heat transfer in the problem of soil freezing in detail. However, when implementing a numerical solution to this problem, it is very difficult to model both a continuous medium and the boundaries between zones at the same time, primarily due to the deficiency of numerical methods used in modeling.

Two different approaches have been formed to ensure the sustainable solution of this kind of problems. *The first approach* takes into account the equations of thermal and material balance at the freezing front, as well as at the other boundaries inside the soil, at which an abrupt change in properties is assumed while maintaining the continuity of the temperature field.

*In the second approach*, the abrupt change in the phase composition and thermophysical properties of the soil at the boundaries of the zones is smoothed out. For example, one of the recognized methods of achieving uniformity is the so-called enthalpy formulation of the Stefan problem [*Ershov, 1999*]. In this formulation, all variables in Eqs. (1) and (2), including temperature, are considered as functions of enthalpy. Enthalpy, in turn, becomes a function of not only the temperature, as in the thawed and frozen parts but also of the coordinates of the freezing front boundary. In this case, the boundaries seem to “get fuzzy”, and the heat balance equation at the front degenerates into a solution of the modified Fourier equation.

Conventionally, models based on the first approach can be called models with a pronounced front, or front models, whereas models based on the second approach are models with a “fuzzy front”, or frontless models.

The main objective of creating models of soil freezing is to solve various geotechnical problems in permafrost areas, in which the stress field created by various construction objects is modeled, and the behavior of the temperature field and possible deformation changes are predicted. Hydrological conditions are taken into account, and a forecast of the influence of various water sources on the deformation of constructions or natural phenomena is given.

Therefore, in addition to direct solution of the thermal problem, soil freezing and heaving necessitate the development of mass transfer equations for water and salt transfer together with thermal energy. In the case of engineering problems solution, it is also necessary to apply the equations of the stress-deformation state of the soil, or modify the equations of heat and mass transfer in such a way that they can take into account the impact of the external load.

When modeling moisture transfer in the frozen part of freezing soils, the researchers relied on the concepts of moisture transfer in thawed soils. This led to two different approaches for solving the problem of moisture transfer. In the first approach, researchers rely on *the dependence of the water flow in soils on the moisture gradient*, applying this approach to frozen soils (wet form). In the alternative approach, *the dependence of the moisture transfer intensity on the pore pressure gradient* is considered.

It should be emphasized that moisture transfer in frozen soil is primarily caused by its inherent temperature gradient. Thus, it is necessary to express the temperature gradient through the pressure gradient inside the pores of the frozen soil to create a single pressure field that determines the flow of water in both thawed and frozen parts. At the same time, different equations of the relationship between pore pressure and temperature are used in the models. This leads to an additional difference between the models in the approach, which considers the dependence of the water transfer intensity on the pore pressure gradient.

## OVERVIEW OF FRONT MODELS

### The Melamed–Feldman model

V.G. Melamed [*1969*] and later G.M. Feldman [*1988*] presented models that take into account the moisture flow when solving the Stefan problem. To solve the problem, V.G. Melamed accepted the presence of moisture transfer in the thawed zone and its absence in the frozen zone, as well as the fact that segregation ice appears only at the freezing boundary. From a mathematical point of view, the formulation of the problem of heat exchange in the freezing soil was faultless, however, the physical statement of the problem of moisture transfer and segregation ice release had a fundamental disadvantage, due to which the mathematical model did not give reliable results.

The fact is that in order to start cryogenic migration and the formation of water flow to the freezing front in the soil, the moisture gradient was used as the driving force of the calculation models, and not the moisture potential gradient (or its equivalent –

the hydraulic pore pressure gradient). Such a physical formulation of the problem for mathematical modeling, as numerous further experimental studies showed, did not correspond to the physical essence of the process of cryogenic migration and frost heaving of soils [*Kudryavtsev et al., 1973; Cheverev et al., 2021*].

### Models based on the concept of the freezing layer

When describing the works of Konrad, Miller, and other foreign authors on the problem of soil freezing based on the consideration of the pore pressure gradient as a driving force and the concept of the existence of a special freezing layer, it should be clarified what they were modeling. In these works, the authors tried to describe the formation of a cryogenic schlieren structure during soil freezing, as well as to take into account the influence of external load on the processes. During freezing, they observed the formation of periodic massive ice lenses located transversely to the thermal current and blocking moisture transfer in the soil. In this case, the rate of change in the thickness of such a lens could be easily measured, and consequently, the amount of water flow entering such a lens could be calculated.

R. Miller proposed the concept of an intermediate (hereinafter, freezing) layer directly behind the freezing front up to the first ice lens. He proved that in the region of temperatures below the freezing point, in some narrow region, there will be such conditions under which part of the pores will be filled with ice, and another part will be supercooled (unfrozen – *Approx. authors*) water, unable to turn into ice due to the small size of the pore itself [*Miller, 1978*]. The temperature gradient in this layer will be proportional to the pressure gradient of the supercooled water in the pores and will determine the amount of water flow coming to the ice lens immediately behind this layer.

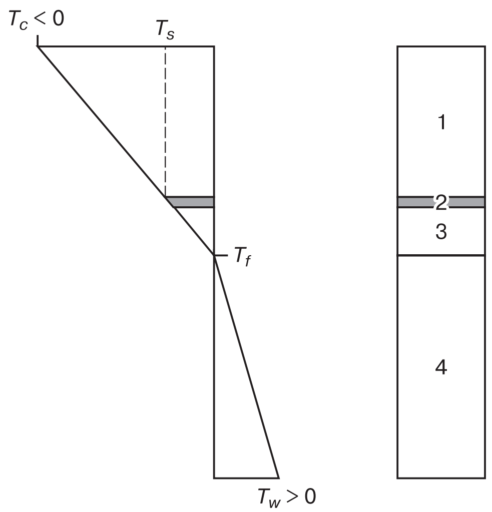


**Fig. 1. Simulation model of frost heaving [*Konrad, Morgenstern, 1980*].**

1 – frozen layer, 2 – ice lens, 3 – freezing layer, 4 – thawed layer. $T_c$ is the temperature at the boundary of the frozen layer, $T_s$ is the temperature at the boundary of the ice lens, $T_f$ is the temperature at the freezing front, and $T_w$ is the temperature at the boundary of the thawed layer.

### The Konrad–Morgenstern model

The authors in [*Konrad, Morgenstern, 1980*] theoretically substantiated and empirically confirmed that the moisture flow in freezing soils is directly proportional to the temperature gradient in the frozen zone. In subsequent papers, the proportionality coefficient between them was named the segregation potential (analogous to the moisture conductivity coefficient). The mass transfer equation proposed by Konrad, can be written as

$$v_u = SP \,\mathrm{grad}\,(T_f), \tag{3}$$

where $SP$ is the segregation potential, m$^4$/(K·s); $v_u$ is the water flow rate (change in the volume of water entering the ice lens per unit of time), m$^3$/s; grad ($T_f$) is the temperature gradient in the frozen zone of the sample, K/m.

As a basis, Konrad used a comparatively simple model, in which two zones were considered: thawed and frozen, separated by a freezing front (Fig. 1).

He took the temperature gradient in the freezing layer to be equal to the temperature gradient in the frozen layer. As a theoretical justification, the author used the Clapeyron–Clausius equation for the connection between the pressure of unfrozen water in the pores of the intermediate layer and the temperature gradient in it. The approach of J.M. Konrad and his co-authors has gained great popularity abroad. On the basis of this empirical concept, for example, the model [*Loranger, 2020*] is constructed, in which heat is added to Eqs. (2) and (3) due to freezing of water, which is supplied by cryogenic migration:

$$\lambda_f \nabla T_- = \lambda_u \nabla T_+ + L_s \frac{\Delta z}{\Delta t} + L_w v,$$

where $\lambda_f$, $\lambda_\mu$ are the thermal conductivity of the frozen and thawed zones, respectively; $\nabla T_-$, $\nabla T_+$ are the temperature gradients of the frozen and thawed zones; $L_s$ is the volumetric latent heat of water freezing at the freezing front; $L_w$ is the volumetric latent heat of water freezing at the contact with ice lens; $\Delta z/\Delta t$ is the speed of movement of the freezing front. The application of such a concept made it possible to describe the freezing process with a slow front movement at small temperature gradients.

### The Gilpin model

In their works, J.M. Konrad and coauthors showed that the water flow calculated through the pressure gradient at the freezing front, is at least an order of magnitude higher than that observed in the physical experiment. To resolve this contradiction, R. Gilpin [*1980*] supposed that the temperature gradient in the freezing layer is different from the gradient in the rest of the frozen soil and considered the heat balance equation directly at the boundary of the freezing layer and the ice lens (Fig. 2):

$$k_f \frac{T_{top}-T_l}{H} - k_p \frac{T_f - T_l}{a} = \frac{L}{v_s} V_H,$$

$$k_p \frac{T_f - T_l}{a} - k_{uf} \frac{T_{bot}-T_f}{x} = \rho_d L \frac{dz}{dt},$$

where $k_f$, $k_p$, $k_{uf}$ are the thermal conductivity coefficients of the frozen and freezing layers and the thawed zone, respectively, W/(m·K); $V_H$ is the growth rate of the ice lens, m/s; $T_{top}$ is the temperature on the cold face, K; $T_l$ is the temperature at the frozen/freezing layers interface (or, in other words, at the boundary of the formation of a continuous layer of ice), K; $T_f$ is the temperature at the freezing front, K; $T_{bot}$ is the temperature at the boundary with the heat source, K; and $v_s$ is the specific volume of ice, m$^3$/kg.

R. Gilpin [*1980*] introduced a significant simplification, taking the temperature profile in the freezing layer to be linear, and did not consider the process of soil freezing in this layer and the interaction of the ice formed in it with the ice lens (meantime, the temperature profile bends smoothly [*Ershov, 1999*]).

The paper [*Bronfenbrener, Bronfenbrener, 2010*] provides an analytical solution to this thermal problem. The authors use the transformation of spatial and temporal coordinates and provide an analytical solution for short and long-term forecasts of freezing.

Within the framework of the Gilpin model, the boundary conditions between the thawed and freezing layers, and the freezing layer and the forming ice lens were only considered. Heat and moisture transfer directly in the freezing layer were not considered. In addition, the problem of the impact of the ice accumulating in the freezing layer on the ice lens behind this layer was not disclosed, though the migration of this ice to the ice lens determines the rate of its growth. The concept of water transfer in this layer was considered in detail in [*O'Neill, Miller, 1985*]. The main ideas of the work are as follows. The freezing layer was considered exclusively between the thawed zone and the first ice lens. If an ice lens appears in the freezing layer, the length of the latter is automatically reduced to the new ice lens. Thus, the condition is fulfilled that there is no ice lens in the freezing zone, and this zone consists exclusively of unfrozen water and ice-cement moving to the new ice lens. This ice-cement is tightly related to the ice lens, i.e., the velocity of movement of ice-cement coincides with the growth rate of ice lens.

The water pressure can be calculated from the generalized Clapeyron–Clausius equation:

$$(P_w - P_0)V_w - (P_i - P_0)V_i = L\frac{\Delta T}{T_0}, \quad (4)$$

where $V_w$, $V_i$ is the specific volume of water and ice, respectively, m$^3$/kg; $P_i$, $P_w$ is the pressure of ice inclusions and unfrozen water, Pa.

As the growth of ice lens is determined by the velocity of migration of ice-cement, the expression of the density of the migration flow through the pressure gradient of the liquid phase in accordance with Darcy's law is not valid. Therefore, as a driving force, the authors consider the gradient of the total pore pressure $P_n$, which determines the rate of ice lens growth, which can be calculated by the formula

$$P_n = \chi P_w + (1-\chi) P_i, \quad (5)$$

where $\chi$ is a parameter depending on the unfrozen water content.

The equations of heat balance on the basis of the Fourier equation are solved with due account for the release of heat upon water freezing and the mass balance, where the water flow is determined by the pressure gradient.

R. Miller's ideas were implemented in a simplified model, which considers a water-saturated system consisting of a continuous porous non-deformable medi-

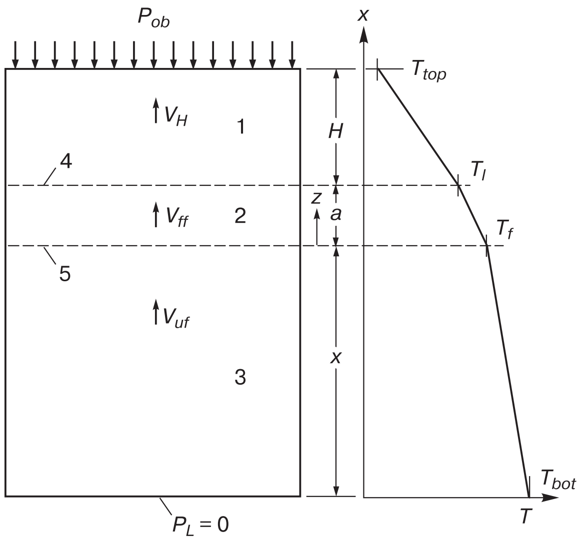


**Fig. 2. Simulation model of frost heaving with determination of values [*Gilpin, 1980*].**

1 – frozen layer, 2 – freezing layer, 3 – unfrozen (thawed) zone, 4 – ice segregation front, 5 – freezing front. $T$ is the temperature, $z$ is the distance, $H$ is the thickness of the frozen layer, $a$ is the thickness of the freezing layer, $T_{bot}$ is the temperature of the warm border, $T_f$ is the temperature at the freezing front, $T_l$ is the temperature at the ice segregation front, $T_{top}$ is the temperature of the cold boundary, $P_{ob}$ is the mechanical pressure at the cold boundary, $P_L$ is the pore pressure at the warm boundary, $V_H$ is the growth rate of the ice lens, $V_{ff}$ is the water flow rate in the freezing layer, and $V_{uf}$ is the water flow rate in the unfrozen zone.

um (capillary porous ceramics) and an ice lens forming on the cold surface of a ceramic cylinder [*Gorelik, Kolunin, 2002*]. The authors conducted an experiment in which they used a sample of porous ceramics, which is suitable for the role of a permanent non-expandable medium. Water was supplied from one side of the ceramic cylinder; on the opposite side, there was a source of cold under the load and a growing ice lens.

In his solution, Ya.B. Gorelik uses the Stefan condition at the freezing front, as well as at the interface of the porous body and the ice lens. Similar to R. Miller, the author uses the intra-pore pressure gradient determined from the generalized Clapeyron–Clausius equation to determine the moisture transfer. The rate of growth of the ice lens was determined based on the velocity of the freezing front movement at the interface of the porous body and ice.

The question of using the Clapeyron–Clausius equation in the freezing zone remains open and disputed. In [*Akagawa et al., 2006*], the authors point out that the formation of an ice lens in the soil actually takes place in a closed system, until the pressure of the ice lens on the soil exceeds the load on the sample and the tensile strength of the soil. After this rupture occurs, the amount of pressure exerted by the ice lens on the pore water drops sharply and the condition of the closed system is disturbed. This leads to a sharp growth of ice lenses.

In [*Ma et al., 2015*] it is proposed that the function of load distribution during the growth of an ice lens in dependence on the degree of pore filling with ice in the freezing zone. In general, the assessment of the influence of the ice lens pressure during its growth is an ambiguous problem.

### The Cheverev–Buldovich model

Equations (4) and (5) are applicable first of all either for a closed, or at least non-expanding medium, which is practically not observed in soils. In [*Cheverev, 1999, 2003a,b, 2004; Ershov, 1999; Cheverev, Safronov, 2012; Cheverev et al., 2013, 2021*], the Edlefsen–Andersen equation is used to describe the dependence of the pore pressure on temperature [*Edlefsen, Anderson, 1943*]:

$$P_w = L\Delta T/(T_0 V_w),$$

where $P_w$ is the equilibrium pressure of the liquid phase of the soil water at the boundary with the solid phase, Pa; $L$ is the specific heat of ice melting, J/kg; $\Delta T$ is a decrease in the freezing temperature of the soil (pore solution) relative to the freezing of unbound water; $T_0$ is the absolute value of the freezing temperature of free water, K; $V_w$ is the specific water volume, m$^3$/kg.

In this case, the density of the water flow in the frozen soil is determined by the temperature gradient [*Cheverev, 1999; Ershov, 1999*]:

$$i_w = \lambda_w(T) K \frac{dT}{dz}, \quad (6)$$

where $i_w$ is the density of the water flow (a vector directed towards the temperature gradient with a value equal to the change in the volume of water per unit of time passing through a unit area), m/day; $\lambda_w(T)$ is the moisture conductivity coefficient of the frozen soil zone, m/day; $K = L/(T_0 V_w)$ is the coefficient of proportionality expressed in meters of water column per degree Kelvin and equal to 120 m/K; and $dT/dz$ is the temperature gradient in the measuring zone.

The density of the water flow in the thawed zone is

$$i_w = \lambda_w KL \frac{T_{nz} - T_\xi}{l - \xi},$$

where $T_{nz}$, $T_\xi$ is the temperature of the beginning of freezing of the soil and the temperature at the freezing front; $\xi$ is the depth of freezing; $l$ is the size of the area of calculations.

In terms of the development of the Cheverev–Buldovich model, a numerical solution of the problem was carried out on the basis of the finite element method and the finite difference method [*Cheverev, Safronov, 2012*]. The solution is based on the calculation of the flow density using Eq. (6). At the same time, for the calculation of $T_\xi$, not an analytical solution of the heat and mass transfer equations is used, as in the Cheverev–Buldovich model, but a numerical solution by tracking moisture at the freezing front when solving the mass balance equations:

$$dT_\xi = \frac{dW_\xi}{dW/dT},$$

where $W_\xi$ is the content of unfrozen water at the freezing front; $\frac{dW}{dT}$ is the derivative of the function of the unfrozen water content from the temperature at point $T_\xi$.

## OVERVIEW OF FRONTLESS MODELS

The system of moisture transfer equations solved with respect to moisture is one of the widespread approaches for modeling freezing processes in soils. The general system of heat and moisture transfer equations in this form was formulated for building capillary-porous materials in the work [*Lykov, Mikhailov, 1963*]:

$$C_p \frac{dT}{dt} = -\operatorname{div}(I_q) - \sum_{i=0}^{4}(H_i T_i - C_i I_{mi} \Delta T),$$

$$\frac{d(\rho_0 W_i)}{dt} = -\operatorname{div}(I_{mi}) + \sum_{i=0}^{4} I_i,$$

$$\frac{d(\rho_0 W_w)}{dt} = -\operatorname{div}\sum_{i=0}^{4} I_{mi},$$

where $C_p$ is the volumetric heat capacity of the soil, consisting of the volume fractions of the heat capacities of the components, J/m$^3$; $\rho_0$ is the density of the soil, kg/m$^3$; the index, for example, $i$ denotes the components of the soil: mineral skeleton, air, water, and ice; the following values of these components are considered: $I_q$ – specific heat flux, J/(kg·m$^2$); $I_{mi}$ – specific heat flux, kg/(m$^2$·s); $I_i$ – discharge capacity, J/(kg·m$^2$); $C_i$ – heat capacity, J/(kg·K); $H_i$ – enthalpy, J/kg; $W_i$, $W_w$ – ice and moisture content, f.u.; the density of the water flow ($I_w$, kg/(m$^2$·s)) in the one-dimensional case is determined by the moisture gradient. Examples of models using this approach are discussed below.

### The Lavrov model

The main difficulty in constructing numerical models without the allocation of thawed and frozen zones is the choice of a variable common to both zones in the system of heat and moisture transfer equations. In [*Lavrov, 2000*] the total soil moisture was taken as such a value. To solve the heat balance equations Fourier equation was considered taking into account phase transitions due to non-frozen water (1). As the driving force for the equation of mass balance in the frozen area, the author expresses the gradient of the unfrozen water content through the ice content and the total moisture content:

$$\frac{dW}{dT}=\frac{d}{dx}\left(D_W\frac{dW}{dx}\right)+\frac{d}{dx}\left(D_P\frac{dP}{dx}\right)+\frac{d}{dx}\left(D_I\frac{dI}{dx}\right),\quad(7)$$

where $W$ is the total moisture content, f.u.; $D_W$, $D_P$, $D_I$ are the corresponding diffusion coefficients due to the gradients of unfrozen water, pressure, and ice; $I$ is the ice content; and $dP/dx$ is the pressure gradient in the system, set by the external pressure.

The heat problem is solved separately from the mass transfer problem. Individual equations defining the behavior of the system as a whole are written for each problem.

### The Danielyan–Yanitsky model

In the work [*Danielyan, Yanitsky, 1983*], the authors used two variables: moisture due to non-freezing water and iciness. They considered the freezing process taking into account the effects of the dynamics of the phase transition of water in the freezing ground. The heat balance equation is based on the Fourier equation, taking into account the freezing of water and, accordingly, ice accumulation; mass transfer is based on equation (7) without taking into account the influence of external pressure. The rate of ice accumulation was determined from the equation

$$\frac{dI}{dt}=a\left(W-W_{nz}\right),$$

where $dI/dt$, the rate of ice accumulation, is the function of moisture content and the direction of the ice melting process or water crystallization; $W_{nz}$ is the content of unfrozen water; and $a$ is the proportionality coefficient obtained by the authors empirically and depending on the rheological properties of frozen soils.

The approach of Danielyan–Yanitsky, apparently, is one of the most interesting from the point of view of ensuring the continuity of the solution of the problem, since the criteria for the formation of ice and the time functions of its appearance are set. It is assumed that ice is "delayed" during crystallization, and this delay sets the ice crystallization process and moisture freezing in a wide area, not just at the interface.

### The Li model

A two-dimensional version of the mass transfer problem, preferably in frozen soil, is presented in [*Li et al., 2013*]. To determine the migration of moisture, the authors express the moisture gradient through the temperature gradient:

$$I_w=\rho_0 D\frac{dW}{dx}=\rho_0 D\frac{dW}{dT}\frac{dT}{dx},$$

where $D$ is the diffusion coefficient of water in the soil.

The authors use the Fourier equation taking into account water migration and phase transitions due to unfrozen water:

$$\left(c\rho+L\rho_w\frac{dW}{dT}\right)\frac{dT}{dt}=\left(k_x+L\rho_w\frac{dW}{dT}D_x\right)\frac{dT}{dx}+$$
$$+\left(k_y+L\rho_w\frac{dW}{dT}D_y\right)\frac{dT}{dy},$$

where $k_x$, $k_y$ are the coefficients of thermal conductivity; $D_x$, $D_y$ are diffusion coefficients; $\rho$ is the soil density; $c$ is the soil heat capacity; and $\rho_w$ is water density.

## POROSITY-BASED MODELS

### The Mikhailovsky–Zhu model

The paper [*Michalowski, Zhu, 2006*] presents a model in which a solution of the thermal problem based on the modified Fourier equation for frozen soil is proposed:

$$C\frac{dT}{dt}-L\frac{d\theta_i}{dt}\rho_i-\nabla\left(\lambda\nabla T\right)=0,$$

where $\nabla$ is the symbol of the gradient; $\theta_i$ is the volume fraction of ice.

The main idea of the model is an attempt to express the main coefficients included in the equation of heat balance and mass balance through the porosity of the soil, since the accumulation of ice changes primarily the spatial geometry of the distribution of mineral particles. So, the volume fraction of ice, $\theta_i$, is calculated by the formula

$$\theta_i=\frac{V_i}{V}=n\left(1-\nu\right),$$

where $V$ is the total volume of the soil; $V_i$ is the volume of ice; $n$ is the porosity of the soil; and $\nu$ is the volume fraction of unfrozen water relative to the total volume of unfrozen water and ice in the soil:

$$\nu = \frac{V_w}{V_w + V_i},$$

where $V_w$ is the volume of unfrozen water; $V_i$ is the volume of ice.

The dependence of porosity on temperature is characterized by a certain maximum and is given by the equation

$$n = n_m \left(\frac{T - T_0}{T_m}\right)^2 \exp\left[1 - \left(\frac{T - T_0}{T_m}\right)^2\right],$$

where $n$ is the porosity of the soil; $n_m$ is the maximum porosity; $T_0$ is the freezing point; $T_m$ is the temperature at maximum porosity; and $T$ is the current temperature.

The equation of material balance is solved with respect to the porosity of the soil:

$$(\rho_i - \rho_s)\frac{dn}{dt} + (\rho_w - \rho_i)\frac{d(n\nu)}{dt} - \rho_w \nabla(\nabla h) = 0,$$

where $\rho_i$, $\rho_w$, $\rho_s$ are the densities of ice, water, and mineral (solid) particles, respectively; $h$ is the water pressure, m.

Among recent works, the work [*Ming et al., 2016*] should be noted, as it suggests a solution based on this approach; it takes into account the stress-strain state of the soil and porosity. The authors consider the general deformation of the soil as the sum of thermal expansion, elastic deformation, and deformation due to changes in the porosity of the soil. The stress state equation is proposed:

$$d\sigma = -dE_s \varepsilon^e = -dE_s \left[\varepsilon - \varepsilon^T - \varepsilon^c\right],$$

where $\sigma$ is tension; $E_s$ is elastic modulus; $\varepsilon^e$ is elastic deformation; $\varepsilon^T$ is deformation due to thermal expansion; $\varepsilon^c$ is creep deformation; and $\varepsilon$ is the total deformation, which can be described by changing the porosity by the formula

$$\varepsilon = \frac{n_0 - n}{1 - n},$$

where $n_0$ is the initial value of porosity, and $n$ is its current value.

In [*Abdalla et al., 2014*], it is also proposed to improve the dependence of the porosity function on temperature for the thawed part and, in addition, to take into account the dependence of thermal conductivity as a function of temperature and porosity with due account for the direction of freezing or thawing.

In [*Li et al., 2018*], a solution to the problem of mass transfer with incomplete water supply is proposed (i.e., the influence of air on the moisture transfer process is taken into account). In addition, in this work, the solution of a mechanical problem is proposed. In contrast to the previous work, the total deformation is considered as the sum of elastic deformation, viscoplastic deformation, and deformation due to frost heaving, which, in turn, depends on porosity. The stress-strain state in the work is solved by representing the total deformation as a sum of deformations due to elastic interaction, heaving deformation, and viscoplastic deformation. Since the stress in the ground, according to [*Li et al., 2018*], arises due to elastic interaction, the elastic component is expressed in terms of the difference between the total deformation, deformation due to heaving, and viscoplastic deformation:

$$\{\Delta\sigma\} = [D_T]\left(\{\Delta\varepsilon\} - \{\Delta\varepsilon_{vp}\} + \{\Delta\varepsilon_{fh}\}\right),$$

where $\sigma$ is the stress; $D_T$ is the modulus of elasticity; $\varepsilon$ is the total deformation; $\varepsilon_{vp}$ is the viscoplastic deformation; $\varepsilon_{fh}$ is the heaving deformation determined by the formula:

$$\Delta\varepsilon_{fh}^V = \theta_i^{t+\Delta t} + \theta_w^{t+\Delta t} - n_s^t,$$

where $\theta_i$, $\theta_w$ are the volume fractions of ice and unfrozen water; and $n_s$ is porosity.

### The Pavlov–Permyakov–Romanov model

The transfer of salt dissolved in the soil water is determined by Fick ‘s law:

$$I_c = D_c \frac{dC}{dx},$$

where $I_c$ is the ion flux density; $C$ is the ion concentration; and $D_c$ is the ion diffusion coefficient.

In the works [*Pavlov, Permyakov, 1983; Permyakov, Romanov, 2000*], a two-dimensional model of salt transfer is considered, which allows modeling the two-dimensional distribution of salt in soils. The model assumes the squeezing of salts from the frozen zone during freezing. For this purpose, a problem is considered that takes into account only conductive heat sources, and salt transfer is determined only by the concentration of salt according to Fick’s law:

$$\frac{dC}{dt} = \frac{d}{dx}\left(D\frac{dC}{dx}\right) + \frac{d}{dy}\left(D\frac{dC}{dy}\right).$$

### The Popov model

The coupling of the moisture transfer process with salt transfer is most fully considered in the work [*Popov, 2006*]. The author takes into account the thermogradient effect, ion adsorption in the diffusion layer, the cross movement of moisture and salt, and salt capture during ice crystallization in the pores. They propose a system of equations of thermal and material balance, including

– the Fourier equation taking into account convective moisture transport

$$C_p \frac{dT}{dt} = -\frac{d}{dx}\left[\lambda \frac{dT}{dx} + c\rho_0 T J_w\right],$$

– the moisture balance equation

$$\frac{dW}{dt}=-\frac{d}{dx}J_w-I_f,$$

– the general equation of moisture and salt balance

$$\frac{d(WC)}{dt}=-\frac{d}{dx}\left[WD_c\frac{dC}{dx}+CJ_w\right]-k_zCI_f-I_a,$$

– the equation for calculating the total water flow

$$J_w=-K\frac{dW}{dx}+K\delta_{CW}\frac{dC}{dx}-K\delta_{TW}\frac{dT}{dx}+V_f,$$

where $I_f$ is the water outflow due to water crystallization; $I_a$ is the outflow of salt due to ion adsorption by the diffusion layer; $J_w$ is the water flow; $K$ is the diffusion coefficient; $K\delta_{CW}\frac{dC}{dx}$ is the component of the water flow due to the cross flow of salt; $K\delta_{TW}\frac{dT}{dx}$ is the component due to the thermogradient effect; the value of $k_zCI_f$ determines the salt capture during ice crystallization; and $V_f$ is the water filtration rate caused by the pressure gradient.

## THERMOMECHANICAL MODELS

A common feature of thermomechanical models is an attempt to describe the joint equations of heat and mass transfer using the equation of the stress-strain state of the soil.

### The Grechishchev model

In [*Grechishchev, 1983*], the soil is considered as a filtration-consolidation medium. To describe the heat balance for the frozen and thawed zones, S.E. Grechishchev used the Fourier equations. The model describes the filtration consolidation equations relating the pressure in the soil to the stress-strain state and the water flow velocity in it.

The model considers the phase equilibrium of the linked small and large pores using the Clapeyron–Clausius equation; for large pores, the stress on the pore ice matrix is taken into account. Temperature fields are determined from this equilibrium. In the model, the equations of phase equilibrium take into account the kinetics of mass transfer and the movement of the freezing front.

### The Razbegin model

V.N. Razbegin solves a heat problem with a freezing boundary, where three zones are distinguished: thawed, frozen, and freezing [*Razbegin, 1983*]. At the same time, the Fourier equation was used to set the heat balance equations for thawed and frozen zones, and for the material balance in the thawed zone, an equation based on the moisture gradient as a driving force, taking into account the thermogradient coefficient (δ) was applied:

$$\frac{dW}{dT}=\frac{d}{dx}\left(a\frac{dW}{dx}+a\delta\frac{dT}{dx}\right),$$

in the zone of phase transitions, taking into account only the thermogradient coefficient

$$\frac{dW}{dT}=\frac{d}{dx}\left(a\delta\frac{dT}{dx}\right).$$

A feature of the author's approach is the generalization of the equations of material and heat balance regarding the case of the occurrence of deformation and stress fields.

Using the thermodynamic approach and the dependence of generalized thermodynamic functions on the strain-stress state, the author presents a system of equations for the combined solution of the strain-stress state, the heat problem, and the mass transfer problem.

## CONCLUSIONS

The models, in which the problems of combined moisture, salt, and heat transfer and the mechanical stresses accompanying this process are solved, are the most perfect models from the point of view of their physical formulation. These models describe the solution of the problems in the most comprehensive way. However, the complexity of these models, primarily in determining the main coefficients included in the equations, sharply limits their practical applicability.

The table shows the typification of the physical formulation of the problem of mathematical models of freezing and heaving of soils, which was compiled by the authors of the article on the basis of the above analysis.

The tasks ***of front models*** can be divided into three groups. In the models of group I used in the works of V.G. Melamed and G.M. Feldman, the moisture gradient is considered as the driving force of moisture transfer. The Stefan problem is solved at the freezing front, and it is assumed that the moisture gradient is determined by the difference in moisture from the initial total soil moisture to the values of the unfrozen water content at the freezing front. Such a solution is not perfect and unambiguous, since it has been experimentally proved that cryogenic migration in the thawed zone can occur in a gradient-free moisture field, but at the same time in a gradient field of pore pressure, which is set by temperature at the phase boundary [*Cheverev, 2004*].

The models based on R. Miller's approach can be combined into Group II. In these models, the freezing soil is divided into three zones: thawed, transitional (freezing), and frozen. In this case, the formation of ice lens is considered at the border of the transitional and frozen zones. The temperature of the freezing front is always fixed and is equal to the freezing temperature of the pore solution, and the temperature at the boundary of the transition and frozen zones is variable and is a function of pore pressure, moisture, and the thermal regime of freezing. A significant disadvantage of this approach is simplification, in which the process of moisture transfer is considered only in the area of the so-called freezing layer located between the freezing front and the first ice lens, and then, behind the ice lens, the process of mass transfer is ignored. At the same time, the work [*Ershov, 1999*] showed the presence of unfrozen water flow behind the ice lens, i.e., it is quite possible to assume the existence of an intermediate layer with ice lenses.

In this case, the value of the pore pressure in the freezing zone is determined from Eq. (4), and at the contact of the freezing zone with the growing ice lens, the pressure on the ice (in Eq. (4)) is equated to the external load. As a significant advantage of this approach, it is necessary to indicate the possibility of solving a number of engineering problems that take into account the influence of external load [*Gorelik, 2010*].

In Group III models, for example in the Cheverev–Buldovich and Cheverev–Safronov models, the

Table 1. **Typification of the physical formulation of the problem in mathematical models of freezing and heaving of soils**

| Group | Model special features | Author |
|---|---|---|
| | *Front models* | |
| I | The solution of the Stefan problem at the freezing front, taking into account convective moisture transfer. The water flow is set by the moisture gradient before the freezing front. The moisture at the freezing front is assumed to be equal to the moisture of the lower limit of plasticity | [*Melamed, 1969; Feldman, 1988*] |
| II | The driving force of cryogenic migration is the gradient of pore equilibrium pressure determined from the generalized Clapeyron–Clausius equation. Models with three zones in the freezing soil: thawed, transitional, frozen. Two boundary conditions with a mobile zone: at the interface of thawed and transitional zone and at the interface of transitional and frozen zones | [*Miller, 1978*] |
| II | Solving the problem of mass transfer to the frozen zone due to the temperature gradient in it. Application of the concept of segregation potential | [*Konrad, Morgenstern, 1980*] |
| II | Ice release at the boundary of the thawed and frozen zones in the form of a continuous ice lens; the Stefan equation at the boundary is replaced by the ice lens growth equation. Freezing is considered only at the boundaries of the intermediate layer. The segregation criterion is formulated from the condition of the pore pressure to the external load equilibrium | [*Gilpin, 1980*] |
| II | Following Gilpin, the segregation criterion is formulated from the condition of equality of pore pressure to external load. An algorithm for calculating layered structures has been developed taking into account the external load, but accounting for the properties of real soils | [*O'Neill, Miller, 1985*] |
| III | Consideration of thawed, transitional (freezing), and frozen zones. Setting the flow of water into the frozen zone through the gradient of the pore pressure of water in the frozen zone based on the Edlefsen–Andersen equations (in potential form). Analytical solution of the problem | [*Ershov, 1999*] |
| II | Further development of the Miller and Gilpin approach in the freezing zone is considered as in Gilpin, but the pore ice is associated with the body of the growing ice lens | [*Gorelik, Kolunin, 2002*] |
| III | Development of the Cheverev–Buldovich model. A numerical solution of the problem with an automatic change in the pressure gradient in the thawed zone depending on the state of freezing, gradients of thawed and frozen zones, and boundary conditions is proposed. The Fourier equation is solved in thawed and frozen zones | [*Cheverev, Sarfonov, 2012*] |
| | *Frontless models* | |
| I | The equations of material balance are solved through the total moisture content and iciness. Accounting for phase transitions due to unfrozen water in Fourier equations | [*Lavrov, 2000*] |
| I | Introduction of relaxation time of ice crystallization and melting. With ice formation in a frozen state | [*Danielyan, Yanitsky, 1983*] |
| II | Modified Fourier equation for thermal balance and material balance equation based on porosity | [*Michailovsky, Zhu, 2006*] |
| I | Two-dimensional solution of the problem only for the frozen state of the soil. Transformation of the water flow rate equation from a moisture gradient to a temperature gradient | [*Li et al., 2013*] |
| I | Solving the problem of joint heat, water, and salt transfer taking into account the cross effects of water transfer, thermogradient effect, the effect of salt adsorption by diffusion layer and salt capture during crystallization | [*Popov, 2006*] |
| III | Thermorheological model. Finding the water flow velocity taking into account changes in the pore pressure gradient. The pore pressure is derived from the equation of the stress-strain state of the soil. The solution of Stefan's problem is found taking into account the changing pore pressure | [*Grechishchev, 1983*] |
| III | The solution of the problem taking into account the deformation-stress state of the soil during freezing. The driving force of migration is the gradient of the equilibrium pore pressure determined by the stress-strain state | [*Razbegin, 1983*] |

transition zone actually corresponds to the zone of limit of the soil shrinkage, and the freezing process is considered primarily in the zone of intense ice release in the temperature range from $T_{bf}$ to $T_{bf}$ –0.6 (where $T_{bf}$ is the freezing point, °C). In these models, the Edlefsen–Andersen equation is used as the equation of the dependence of pore pressure in the soil on temperature; in fact, this is a modified Clapeyron–Clausius equation, where it is assumed that ice formed during freezing does not apply pressure on water.

This statement is true if the freezing zone, namely, the pore pressure in it, is not affected by the external load factor or hydraulic pressure of a different genesis. The consideration of this factor is implemented in the work [*Cheverev, 2004, p. 16*]. One of the advantages of this approach is the possibility of taking into account hydrogeological conditions and external load, under which it is necessary to simulate the process of deformation of the heaving of freezing soil in the presence of additional pressure from its thawed zone.

A characteristic advantage of ***the frontless models*** currently available is the possibility of two- and three-dimensional modeling of freezing processes with an uneven distribution of both heat and water sources in space and time. Recently, there has been a tendency to expand the scope of application of models of this type by taking into account the influence of mechanical load, for example, in the works [*Ming et al., 2016; Li et al., 2018*].

Frontless models can be conditionally subdivided into three groups. The models constructed on the basis of A.V. Lykov's equations belong to group I. The principle of moisture transfer in these models is based on the moisture gradient. The disadvantage of this approach is the requirement for the presence of moisture gradient in the thawed zone. This is often associated with the difficulty of modeling the freezing of dense soils with the moisture at the shrinkage limit. This is especially characteristic of silty sands and loamy sands. It is also important to take into account the relaxation nature of rheological deformation and, consequently, changes in soil density, even if the zone has internal sources of moisture and the moisture gradient is well defined. In addition, difficulties arise with the use of the moisture content for frozen soils due to the non-monotonous (with an extremum) dependence of the diffusion coefficient of unfrozen water on temperature. There is also the difficulty of physical verification of the model due to the consideration of ice fields when solving the equations of the material balance. The situation becomes even more complicated with salt removal in soils due to the known overlapping effects of mutually intersecting thermal and material flows. It should be emphasized that this method requires complex methods of monitoring the dynamics of moisture and ice content fields in frozen and thawed soils.

Group II should include models in which the equations of material and heat balance are solved by introducing porosity as a variable. An additional advantage of this approach is the possibility of solving a mechanical problem. The main disadvantage of the approach is the difficulty of obtaining the appropriate coefficients in the equations for the practical application of these models, because, in addition to ice and water in frozen soils, air also has a significant effect on porosity. Despite the number of works, in which an attempt was made to theoretically account for this influence, significant refinement of practical methods for determining the appropriate amendments to such an influence is required.

Group III should include models in which, in addition to heat and material balance, problems related to the deformation-stress state of soils are solved. This approach is, in our opinion, the most correct and solves a number of problems for Group I tasks. However, it aggravates the already complex physical verification of the model and the determination of the main parameters specified in the equations of the stress-deformation state, phase equilibrium, and heat and material balance.

This greatly narrows the scope of the actual applicability of these models in pactice at the present time.

***Funding.*** *This study was carried out with financial support of the Research Institute of Transneft JSC (contract no. 4220 P/20-511/2015 dated November 19, 2015).*

## References

Abdalla B., Fan C., Mckinnon C. et al., 2014. Extended porosity rate function for frost heave. In: 33rd Int. Conf. Ocean, Offshore and Arctic Engineering. Vol. 10: Polar and Arctic Science and Technology (San Francisco, California, USA, June 8–13, 2014). New York, Publ. ASME, V010T07A046. DOI: 10.1115/OMAE2014-24221

Akagawa S., Satoh M., Kanie S. et al., 2006. Effect of tensile strength on ice lens initiation temperature. In: 13th Int. Conf. Cold Regions Engineering (Orono, Maine, United States, July 23–26, 2006). Reston, VA, United States, Publ. ASCE, p. 1–12. DOI: 10.1061/40836(210)43

Bronfenbrener L., Bronfenbrener R., 2010. Modeling frost heave in freezing soils. *Cold Reg. Sci. Technol.*, **61**, 43–64.

Brovka G.P., 1991. Heat and Mass Transfer in Natural Dispersed Systems during Freezing. Minsk, Nauka i Tekhnika, 126 p. (in Russian).

Cheverev V.G., 1999. Physicochemical theory of the formation of mass-exchange and thermal properties of cryogenic soils. Author's abstract of Doctoral (Geol-Mineral.) Theses. Moscow, 41 p. (in Russian).

Cheverev V.G., 2003a. Classification of water bond forms in the frozen fine-grained soils. *Kriosfera Zemli*, **VII** (3), 31–40 (in Russian).

Cheverev V.G., 2003b. Properties of bound water in cryogenic grounds (analytic review). *Kriosfera Zemli,* **VII** (2), 30–41 (in Russian).

Cheverev V.G., 2004. Nature of Cryogenic Properties of Soils. Moscow, Nauchn. mir, 234 p. (in Russian).

Cheverev V.G., Brushkov A.V., Polovkov S.A., Pokrovskaya E.A., Safronov E.V., 2021. Analysis of concepts on the mechanism of cryogenic water migration in freezing soils. *Earth's Cryosphere,* **XXV** (5), 3–12.

Cheverev V.G., Burnaev R.S., Gagarin V.E., Safronov E.V., 2013. Influence of the external pressure on the degree of frosty heaving of clay soils. *Earth's Cryosphere,* **XVII** (4), 57–62.

Cheverev V.G., Safronov E.V., 2012. Mathematical modeling of soil freezing with visualization of the forming cryogenic structure. In: 10th Int. Conf. Permafrost (TICOP) Resources and Risks of Permafrost Regions in a Changing World (Salekhard, 2012). Tyumen, Pechatnik, vol. 3, p. 561–564 (in Russian).

Danielyan Yu.S., Yanitsky P.A., 1983. Features of nonequilibrium redistribution of moisture during freezing and thawing of dispersed soils. *Inzhen-Fizich. Zh.*, **44** (1), 91–98 (in Russian).

Edlefsen N.E., Anderson A.B.C., 1943. Thermodynamics of Soil Moisture. Hilgardia, p. 31–298.

Ershov E.D. (Ed.), 1999. Fundamentals of Geocryology. Part 5. Engineering Geocryology. Moscow, Izd. Mosk. Gos. Univ., 526 p. (in Russian).

Feldman G.M., 1988. The Movement of Water in Thawed and Freezing Soils. Novosibirsk, Nauka, 256 p. (in Russian).

Getz I.G., Meirmanov A.M., 2000. Generalized solution of the Stefan problem with kinetic supercooling. *Sib. Zh. Industial. Matemat.* **3** (1), 66–86 (in Russian).

Gilpin R., 1980. A model for the prediction of ice lensing and frost heave in soils. *Water Resour. Res.*, **16** (5), 918–930.

Gorelik Ya.B., 2010. On the calculation methods of the engineering construction displacements caused by freezing layer frost heave process. *Kriosfera Zemli,* **XIV** (1), 50–62 (in Russian).

Gorelik Ya.B., Kolunin V.S., 2002. Physics and Modeling of Cryogenic Processes in the Lithosphere. Novosibirsk, Publ. House SB RAS, Geo Branch, 317 p. (in Russian).

Grechishchev S.E., 1983. On the basics of thermorheology of cryogenic rocks. In: Problems of Geocryology. Moscow, Nauka, p. 90–100 (in Russian).

Konrad J.M., Morgenstern N.R., 1980. Mechanistic theory of ice lens formation in fine-grained soils. *Can. Geotech. J.*, **17**, 473–486.

Kudryavtsev V.A. (Ed.), 1978. General Geocryology. Moscow, Izd. Mosk. Gos. Univ., 464 p. (in Russian).

Kudryavtsev V.A., Ershov E.D., Cheverev V.G., 1973. Moisture transfer in frozen soils. *Byull. Mosk. Gos. Univ., Ser. Geol.*, 5, 26–34 (in Russian).

Lavrov S.A., 2000. Processes of heat and moisture transfer in the soil and snow cover. Author's abstract of Doctoral (Technical Sci.) theses. St. Petersburg, 40 p. (in Russian).

Li Z., Liu S., Feng Y. et al., 2013. Numerical study on the effect of frost heave prevention with different canal lining structures in seasonally frozen ground regions. *Cold Reg. Sci. Technol.,* **85**, 242–249.

Li S., Zhang M., Pei W. et al., 2018. Experimental and numerical simulations on heat-water-mechanics interaction mechanism in a freezing soil. *Appl. Thermal Engin.*, **132**, 209–220.

Loranger B., 2020. Laboratory investigation of frost susceptibility of crushed rock aggregates and field assessment of frost heave and frost depth. Thesis for the Degree of Ph. D. Norway, Trondheim, 189 p.

Lykov A.B., Mikhailov Yu.A., 1963. Theory of Heat and Mass Transfer. Moscow; Leningrad, Gosenergoizdat, 536 p. (in Russian).

Ma W., Zhang L., Yang C.-S., 2015. Discussion of the applicability of the generalized Clausius–Clapeyron equation and the frozen fringe process. *Earth-Sci. Rev.*, 142, 47–59.

Melamed V.G., 1969. Solution of the problem of freezing of fine-grained soils taking into account the migration of water to the freezing front. In: Merzlotn. Issled. (Mosk. Gos. Univ.), iss. 9, 90–100 (in Russian).

Michalowski R.L., Zhu M., 2006. Modelling of freezing in frost-susceptible soils. *Comp. Assisted Mechanics Engineer. Sci.*, 13, 613–625.

Miller R.D., 1978. Frost heaving in non-colloidal soils. In: 3rd Int. Conf. Permafrost (Edmonton, Alberta, Canada, July 10–13, 1978) Proc. Ottawa, Natl. Res. Council of Canada, vol. 1, p. 708–713.

Ming F., Zhang Y., Dongging Li., 2016. Experimental and theoretical investigations into the formation of ice lenses in deformable porous media. *Geosciences J.*, **20**, 667–679.

O'Neill K., Miller R., 1985. Exploration of a rigid ice model of frost heave. *Water Resour. Res.*, **21** (3), 281–296, DOI: 10.1029/WR021i003p00281.

Pavlov A.R., Permyakov P.P., 1983. Mathematical model and algorithms for calculating heat and mass transfer on a computer during soil freezing. *Inzhenern.-Fizich. Zh.,* **44** (2), 311–316 (in Russian).

Permyakov P.P., Romanov P.G., 2000. Heat and Salt Transfer in Frozen Unsaturated Soils, Yakutsk, Publ. House SO RAN, 128 p. (in Russian).

Popov V.I., 2006. Mathematical modeling of heat and mass transfer in rocks using a phase equilibrium diagram. Cand. Sci. (Technical Sci.) Diss., Yakutsk, 120 p. (in Russian).

Razbegin V.N., 1983. Thermodynamic Aspects of Mechanics of Frozen Soils. Moscow, Nauka, 104 p. (in Russian).

Tikhonov A.N., Samarsky A.A., 1999. Equations of Mathematical Physics. Moscow, Nauka, 196 p. (in Russian).

*Received July 3, 2021*
*Revised January 17, 2022*
*Accepted April 19, 2022*

*Translated by S.B. Sokolov*