

## SNOW COVER AND GLACIERS

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PARAMETERIZATION OF DAILY AMPLITUDES OF SURFACE AIR TEMPERATURE  
IN GREENLAND FOR APPLICATION IN MASS BALANCE CALCULATIONSA.P. Nevecherja<sup>1,2</sup>, O.O. Rybak<sup>1,3</sup><sup>1</sup> Institute of Numerical Mathematics, RAS, 8, Gubkina str., Moscow, 119333, Russia<sup>2</sup> Kuban State University, 149, Stavropolskaya str., Krasnodar, 350040, Russia; [artiom1989@mail.ru](mailto:artiom1989@mail.ru)<sup>3</sup> Sochi Research Centre, RAS, 8a, Theatralnaya str., Sochi, 354000, Russia; [orybak@vub.ac.be](mailto:orybak@vub.ac.be)

Mathematical modeling of melting (ablation) on the surface of the Greenland ice sheet is one of the challenging tasks of the modern glaciology. Normally, the ablation rate is evaluated by using one of the two methods – either by the temperature index method or by the energy balance method (or by combination of both). The pitfall of either method is realistic approximation of the daily amplitudes of surface air temperatures and of the mean square errors of surface air temperatures. In the paper, an approach is proposed for approximating the equations of both characteristics. Final approximating models allowed us to establish the dependence of surface air temperature daily amplitudes on time and on the absolute height above the sea level.

*Greenland, climate, statistical methods, ice sheet, surface air temperature, ablation, mass balance*

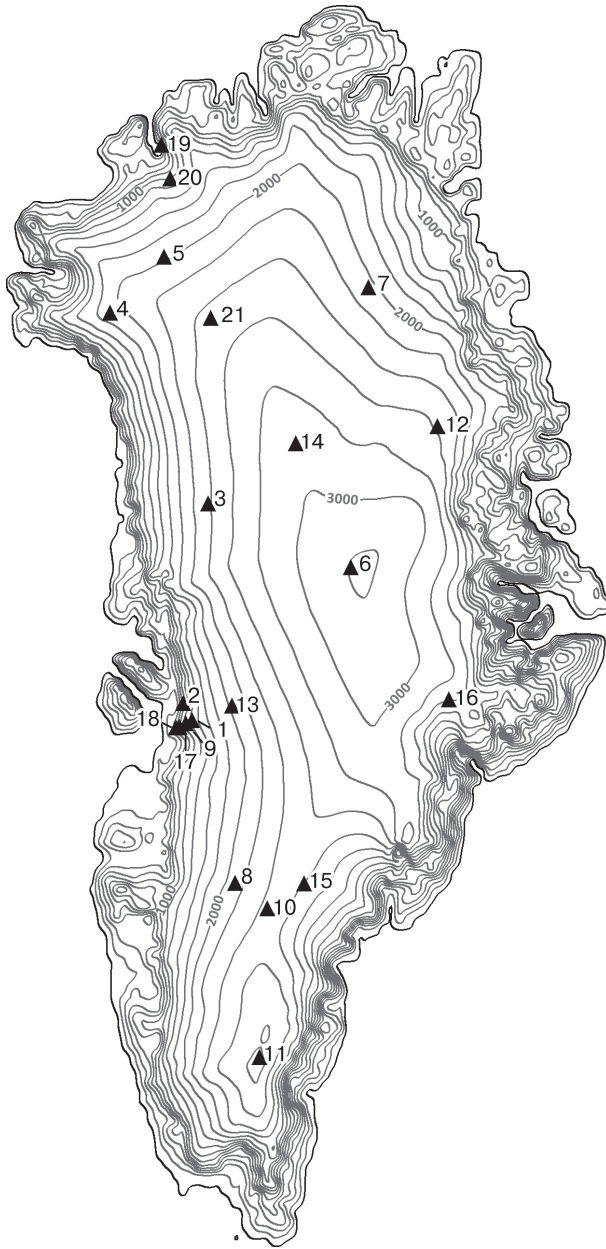
## INTRODUCTION

Methods of mathematical modeling have become widespread in glaciological studies. Ice sheet dynamics is one of the areas of its application. The structure of a modern mathematical model presupposes the presence of a block describing the mass balance on the surface of an ice sheet [Rybak, 2008] which has an incoming (accumulation) and outgoing (ablation) parts. While the outgoing part is represented in Antarctic mainly by the process of sublimation [van de Berg et al., 2005], in Greenland melting plays the major role [Ettema et al., 2010]. To calculate the melting rate, two major approaches are applied in modern mathematical models: the temperature index method, or the positive degree-day method, PDDM [Braithwaite, 1995]) and the energy balance method, EBM, in which the amount of melted water is determined by calculating the amount of accessible energy consumed by melting [Oerlemans, 1991]. The advantage of the PDDM is in its relative simplicity and direct empirical association with the air temperature. The pitfalls consist in significant scatter of the values of the empirically determined dependence of snow or ice melting on air temperature and in rather arbitrary daily and annual variability of the surface air temperature [Rogozhina and Rau, 2014], as well as variations of their statistical characteristics in the case when simulated or modeled mean annual temperature is used as the basic information source. The advantage of the EBM includes its strict physical sustainability, and its disadvantage is mainly related to the complexity of formalizing the computations of certain

key variables, in particular, of the surface albedo. In some cases, the calculation based on the EBM also depends on the way the daily variability of the surface air temperature is described [Rybak et al., 2016].

Despite the growing number of automatic weather stations (AWS) in Greenland, their network is still rather sparse, especially in the inland regions (Fig. 1; Table 1). Therefore, for the purpose of mathematical modeling, different parametrizations of the field of the surface air temperature (SAT) are applied, depending on the latitude, longitude and absolute height of a location [Fausto et al., 2009]. As a rule, such parametrizations are built for mean annual or mean July values of the SAT, whereas the SAT values for the remaining months or days are reconstructed based on the assumption that the annual variability of SAT is a simple harmonic function. More detailed time discretization in mathematical ablation models is achieved by using statistical methods [Zweck and Huybrechts, 2005]. The problem is that the approaches used are based on the hypothesis about the Gaussian distribution of SAT anomalies. Parameters of distribution applied in calculations, primarily the mean square errors of daily mean surface air temperatures, are determined somewhat arbitrarily, which reduces reliability of the results obtained [Rogozhina and Rau, 2014].

In this study, we present the results of statistical analysis of the SAT series obtained by temperature measurements of the Greenland ice sheet made at AWS in Greenland and propose respective approxi-



**Fig. 1. Location of automatic weather stations in Greenland (Table 1).**

ating equations, which may be used for energy- and mass-balance calculations.

### 1. The set-up of the problem and original data

Generally, the problem solved in this study consists in following. If somehow (for example, as a result of mathematical modeling) the field of the mean daily SAT is set for the territory of the Greenland ice sheet, how can the minimum and maximum SAT values be calculated during a day, using meteorological observation data? Although the records of meteorological

**Table 1. Coordinates and the absolute heights of the location of automatic weather stations in Greenland from [Steffen and Box, 2001]**

Number	Station	Observation years	Coordinates		Elevation, m above sea level
			N latitude	W longitude	
1	Swiss Camp	1996–2014	69.56	49.33	1176
2	Crawford Point1	1995–2012	69.88	50.00	2022
3	NASA-U	1995–2014	73.84	49.51	2334
4	GITS	1995–2014	77.14	61.04	1869
5	Humboldt	1995–2014	78.53	56.83	1995
6	Summit	1996–2014	72.58	38.51	3199
7	Tunu-N	1996–2014	78.02	33.98	2052
8	DYE-2	1996–2014	66.48	46.28	2099
9	JAR 1	1997–2014	69.50	49.70	932
10	Saddle	1997–2014	66.00	44.50	2467
11	South Dome	1997–2014	63.15	44.82	2901
12	NASA-E	1997–2014	75.00	30.00	2614
13	Crawford P2	1996–2001	69.91	46.85	1990
14	NGRIP	2002–2010	75.10	42.33	2941
15	NASA-SE	1998–2014	66.48	42.50	2373
16	KAR	1999–2001	69.70	33.01	2579
17	JAR 2	1999–2013	69.41	50.09	507
18	JAR3	2000–2004	69.39	50.31	283
19	Petermann Gl.	2002–2006	80.68	60.29	37
20	Peterman ELA	2003–2014	80.08	58.07	965
21	NEEM	2006–2014	77.50	50.87	2454

observation data obtained from the few AWS in Greenland are rather short, they allow generalization, in the form of approximating equations, the dependence of the maximum and minimum mean annual SAT values (and respectively, of the SAT amplitudes) on the duration of a calendar day and the absolute elevation above sea level.

As initial data, hourly measurements of SAT at 21 AWS were used ( $j = 1, \dots, 21$ ) (Table 1) [Steffen and Box, 2001]. The stations are located not quite uniformly on the territory of Greenland (Fig. 1), which naturally can influence the final results of calculations. Duration of the records is also different. Nevertheless, the stations cover the greater part of the territory of Greenland and, quite importantly, the entire range of altitudes – from the sea coast to the highest inland point.

### 2. Parametrization of daily maximum and minimum values of SAT

Some formal definitions have to be explained. The mean daily SAT anomaly  $y_j(\lambda)$  obtained at each  $j$ -th AWS (Table 1) is understood as deviation of the mean daily SAT value from the respective mean annual value:

$$y_j(\lambda) = c_j(\lambda) - a_j. \quad (1)$$

Here  $a_j$  is the mean annual value of SAT, the result of averaging the original observation data for all the days of the year  $\lambda=1, \dots, \Lambda$  ( $\Lambda = 365$  or  $366$  days of the year), for all the hourly values  $t=1, \dots, T$  ( $T=24$ ) and for all the observation years  $r=1, \dots, M_j$ ,  $M_j$  – the number of years of observation  $\lambda$  obtained at the  $j$ -th AWS:

$$a_j = \frac{1}{T\Lambda M_j} \sum_{r=1}^{M_j} \sum_{\lambda=1}^{\Lambda} \sum_{t=1}^T x_{j,r,\lambda,t}, \quad (2)$$

where  $x_{j,r,\lambda,t}$  are the original observation data over SAT at the  $j$ -th AWS in year  $r$ , on day of the year  $\lambda$  and at hour  $t$ . In (1)  $c_j(\lambda)$  is the multi-year average of the daily mean value of SAT on day  $\lambda$  obtained at the  $j$ -th AWS:

$$c_j(\lambda) = \frac{1}{TM_j} \sum_{r=1}^{M_j} \sum_{t=1}^T x_{j,r,\lambda,t}. \quad (3)$$

In other words, in (1)–(3) the mean annual seasonal behavior of SAT is formally determined for the  $j$ -th AWS with daily resolution. In order to solve the problem set in Section 1, it is necessary to approximate the mean annual minimum and maximum deviations from the mean daily values. We shall name them mean maxima and mean minima:  $c_{\lambda,j}^{(\min)} = \min_t c_{j,\lambda,t}$  и  $c_{\lambda,j}^{(\max)} = \max_t c_{j,\lambda,t}$ , meaning these are not absolute values but deviations from the mean daily values. Fig. 2, *a, b* demonstrates examples of their distribution  $c_{\lambda,j}^{(\min)}$  and  $c_{\lambda,j}^{(\max)}$  at the *Tunu-N* and *Petermann Gl* AWS, respectively, located in the inland region and on the coast, accordingly ( $j = 7$  and  $19$ ).

Let us calculate the approximating functions for  $c_{\lambda,j}^{(\min)}$  and  $c_{\lambda,j}^{(\max)}$  for each AWS. The annual variability of  $c_{\lambda,j}^{(\min)}$  and  $c_{\lambda,j}^{(\max)}$  includes two local minima and two local maxima, which is typical of all AWS (Fig. 2). Hence, it is reasonable to approximate the time series  $c_{\lambda,j}^{(\min)}$  and  $c_{\lambda,j}^{(\max)}$  by a periodic function

by a polynomial function of the 5<sup>th</sup> order. Preliminary fitting of the time series by using the gradient projection method [Levitin and Polyak, 1966] showed the values  $c_{\lambda,j}^{(\min)}$  to deviate least from the polynomial function

$$f_j^{(\min)}(t) = \sum_{i=0}^5 b_{ij}' t^i, \quad j = \overline{1, n}. \quad (4)$$

The values of  $c_{\lambda,j}^{(\max)}$  are best approximated by the periodic function

$$f_j^{(\max)}(t) = b_{0j}'' + \sum_{i=1}^4 b_{2i-1,j}'' \sin\left(\frac{ti\pi}{365} - b_{2i,j}''\right), \quad (5)$$

$$b_{ij}'' \geq 0, \quad b_{2k,j}'' \leq 2\pi, \quad i = \overline{0, 8}, \quad k = \overline{1, 4}, \quad j = \overline{1, n},$$

where  $n$  is the number of stations.

To obtain the parameters  $b_{ij}'$  ( $i = \overline{0, 5}, j = \overline{1, n}$ ) and  $b_{ij}''$  ( $i = \overline{0, 8}, j = \overline{1, n}$ ), we shall use the least squares method to solve the following optimization problems:

$$F^{(\min)}(b') = \sum_{t=1}^{365} \left( c_{t,j}^{(\min)} - \sum_{i=0}^5 b_{ij}' t^i \right)^2 \rightarrow \min; \quad (6)$$

$$F^{(\max)}(b'') = \sum_{t=1}^{365} \left[ c_{t,j}^{(\max)} - b_{0j}'' - \sum_{i=1}^4 b_{2i-1,j}'' \sin\left(\frac{ti\pi}{365} - b_{2i,j}''\right) \right]^2 \rightarrow \min,$$

$$b_{ij}'' \geq 0, \quad b_{2k,j}'' \leq 2\pi, \quad i = \overline{0, 8}, \quad k = \overline{1, 4}, \quad j = \overline{1, n}. \quad (7)$$

To obtain the coefficients in (7), we shall use the method of gradient projection. The results of the calculations are summarized in Tables 2 and 3. Thus, resulting from solution of problems (6) and (7), we obtain 42 parametric equations  $c_{\lambda,j}^{(\min)}$  and  $c_{\lambda,j}^{(\max)}$  for each AWS (21 equations for the mean minimum and maximum sets of SAT values each).

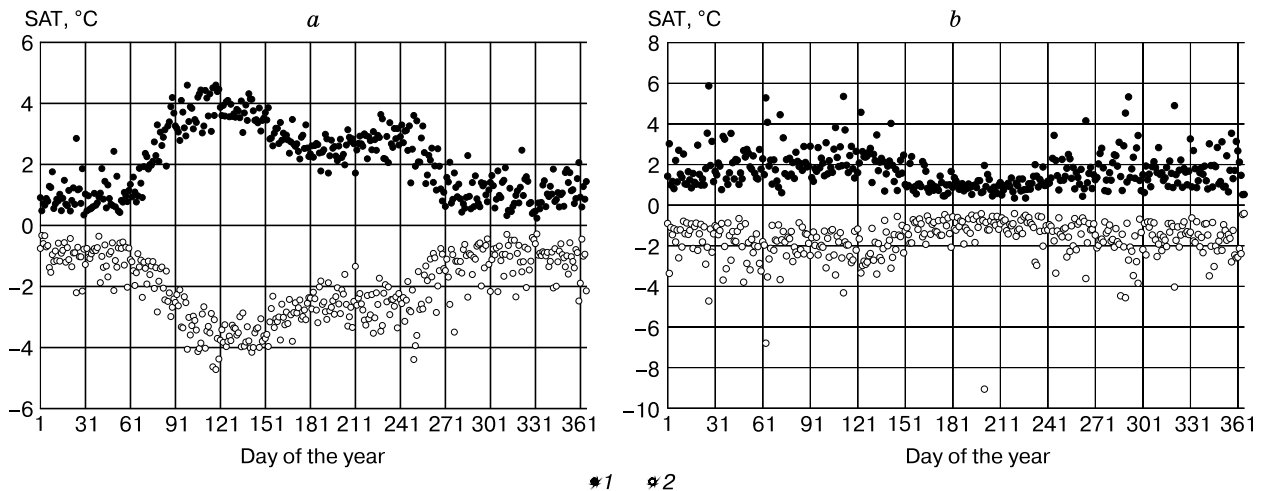


Fig. 2. Mean annual SAT maxima (1) and minima (2) at *Tunu-N* (a) and *Petermann Gl* stations (b).

Table 2. Estimated values of coefficients of parametric equations for approximation of mean SAT minima

$j$	$b'_{0j}$	$b'_{1j}$	$b'_{2j}$	$b'_{3j}$	$b'_{4j}$	$b'_{5j}$
1	-0.92	$1.57 \cdot 10^{-2}$	$-7.6 \cdot 10^{-4}$	$6.63 \cdot 10^{-6}$	$-2.1 \cdot 10^{-8}$	$2.28 \cdot 10^{-11}$
2	-1.44	$3.09 \cdot 10^{-2}$	$-9.1 \cdot 10^{-4}$	$6.31 \cdot 10^{-6}$	$-1.7 \cdot 10^{-8}$	$1.5 \cdot 10^{-11}$
3	-1.66	$1.56 \cdot 10^{-2}$	$-7.8 \cdot 10^{-4}$	$6.32 \cdot 10^{-6}$	$-1.9 \cdot 10^{-8}$	$1.86 \cdot 10^{-11}$
4	-1.59	$2.72 \cdot 10^{-2}$	$-7.9 \cdot 10^{-4}$	$6.04 \cdot 10^{-6}$	$-1.8 \cdot 10^{-8}$	$1.78 \cdot 10^{-11}$
5	-1.40	$3.14 \cdot 10^{-2}$	$-8.7 \cdot 10^{-4}$	$5.95 \cdot 10^{-6}$	$-1.6 \cdot 10^{-8}$	$1.41 \cdot 10^{-11}$
6	-1.69	$5.76 \cdot 10^{-2}$	$-2.08 \cdot 10^{-3}$	$1.65 \cdot 10^{-5}$	$-4.9 \cdot 10^{-8}$	$5.03 \cdot 10^{-11}$
7	-1.01	$2.91 \cdot 10^{-2}$	$-1.02 \cdot 10^{-3}$	$7.36 \cdot 10^{-6}$	$-2.0 \cdot 10^{-8}$	$1.82 \cdot 10^{-11}$
8	-1.16	$3.94 \cdot 10^{-2}$	$-1.41 \cdot 10^{-3}$	$1.08 \cdot 10^{-5}$	$-3.1 \cdot 10^{-8}$	$3.16 \cdot 10^{-11}$
9	-0.79	$1.97 \cdot 10^{-2}$	$-7.9 \cdot 10^{-4}$	$6.9 \cdot 10^{-6}$	$-2.2 \cdot 10^{-8}$	$2.47 \cdot 10^{-11}$
10	-1.48	$2.61 \cdot 10^{-2}$	$-1.07 \cdot 10^{-3}$	$8.27 \cdot 10^{-6}$	$-2.4 \cdot 10^{-8}$	$2.35 \cdot 10^{-11}$
11	-0.87	$-1.45 \cdot 10^{-2}$	$-3.7 \cdot 10^{-4}$	$4.04 \cdot 10^{-6}$	$-1.3 \cdot 10^{-8}$	$1.43 \cdot 10^{-11}$
12	-1.31	$2.41 \cdot 10^{-2}$	$-1.17 \cdot 10^{-3}$	$9.43 \cdot 10^{-6}$	$-2.8 \cdot 10^{-8}$	$2.76 \cdot 10^{-11}$
13	-2.22	$1.93 \cdot 10^{-2}$	$-7.3 \cdot 10^{-4}$	$5.79 \cdot 10^{-6}$	$-1.7 \cdot 10^{-8}$	$1.81 \cdot 10^{-11}$
14	-1.78	$2.62 \cdot 10^{-2}$	$-9.3 \cdot 10^{-4}$	$7.34 \cdot 10^{-6}$	$-2.2 \cdot 10^{-8}$	$2.27 \cdot 10^{-11}$
15	-1.49	$3.16 \cdot 10^{-2}$	$-1.14 \cdot 10^{-3}$	$8.29 \cdot 10^{-6}$	$-2.2 \cdot 10^{-8}$	$2.1 \cdot 10^{-11}$
16	-3.17	$1.46 \cdot 10^{-2}$	$-7.0 \cdot 10^{-4}$	$5.47 \cdot 10^{-6}$	$-1.6 \cdot 10^{-8}$	$1.56 \cdot 10^{-11}$
17	-1.34	$9.66 \cdot 10^{-3}$	$-3.4 \cdot 10^{-4}$	$3.29 \cdot 10^{-6}$	$-1.2 \cdot 10^{-8}$	$1.35 \cdot 10^{-11}$
18	-1.27	$-2.14 \cdot 10^{-3}$	$-3.1 \cdot 10^{-4}$	$3.89 \cdot 10^{-6}$	$-1.5 \cdot 10^{-8}$	$1.82 \cdot 10^{-11}$
19	-1.23	$-2.86 \cdot 10^{-2}$	$2.24 \cdot 10^{-4}$	$8.16 \cdot 10^{-8}$	$-3.8 \cdot 10^{-9}$	$6.76 \cdot 10^{-12}$
20	-1.85	$2.86 \cdot 10^{-2}$	$-6.9 \cdot 10^{-4}$	$5.97 \cdot 10^{-6}$	$-2.0 \cdot 10^{-8}$	$2.39 \cdot 10^{-11}$
21	-2.18	$4.05 \cdot 10^{-2}$	$-1.38 \cdot 10^{-4}$	$1.11 \cdot 10^{-5}$	$-3.4 \cdot 10^{-8}$	$3.6 \cdot 10^{-11}$

Table 3. Estimated values of the factors of parametric equations for approximation of mean SAT maxima

$j$	$b''_{0j}$	$b''_{1j}$	$b''_{2j}$	$b''_{3j}$	$b''_{4j}$	$b''_{5j}$	$b''_{6j}$	$b''_{7j}$	$b''_{8j}$
1	1.77	0	0.05	0.67	0	0.66	3.14	0.97	1.86
2	2.47	0	0.16	0.31	0.23	1.30	2.92	0.84	1.49
3	2.49	0	0	1.25	0.83	0	6.28	0.39	1.92
4	1.89	0	0.26	0.67	0.50	0	5.54	0.40	2.32
5	2.16	0	0.16	0.32	0	1.11	3.00	0.76	1.72
6	3.20	0	0.20	1.78	0.78	0	6.28	0.90	1.98
7	2.42	0	0	0.42	0	1.61	2.96	1.14	1.60
8	0.72	2.88	0	0	0	0	6.28	0.49	2.21
9	1.40	0	0.19	0.53	0	0.43	3.17	0.77	1.97
10	2.70	0	0.05	1.26	0.98	0	6.28	0.61	1.82
11	1.48	1.87	0.14	1.05	0.03	0	3.87	0.60	1.33
12	8.74	9.56	3.07	5.39	1.28	0	6.28	1.58	1.54
13	2.15	0.76	0	0.70	0.44	0	1.23	0	4.83
14	2.60	0	0.12	0.40	0	0.84	3.03	0.84	1.70
15	2.81	0	0.14	1.53	1.06	0	6.28	0.50	1.76
16	3.49	0.33	0	0.90	0.28	0.55	3.67	0.48	2.60
17	1.14	0	0	0.40	0	0	2.18	0.22	2.43
18	1.43	0.02	0.15	0.57	0	0.20	6.28	0.30	2.36
19	1.69	0	0	0	1.17	0	2.57	0.40	1.98
20	1.39	0	0.13	0.35	0	0.48	5.90	0.46	3.56
21	2.84	0	0.14	1.10	0.52	0	6.28	0.60	2.04

The next step will consist in generalization of expressions (4) and (5) for the obtained above  $f_{gen}^{(min)}(t, h)$  and  $f_{gen}^{(max)}(t, h)$ . Whereas (4) and (5) were used for calculating the mean minimum and maximum SAT values only in the area of AWS,  $f_{gen}^{(min)}(t, h)$  and  $f_{gen}^{(max)}(t, h)$  must depend exclusively on the absolute elevation  $h$  and time  $t$ . Thus, the parameters of mean SAT minima and maxima of these functions may be calculated everywhere for Greenland:

$$f_{gen}^{(min)}(t, h) = \sum_{i=1}^5 \tilde{b}'_i t^i + \phi^{(min)}(h); \quad (8)$$

$$f_{gen}^{(max)}(t, h) = \sum_{i=1}^4 \tilde{b}''_{2i-1} \sin\left(\frac{t\pi}{365} - \tilde{b}''_{2i}\right) + \phi^{(max)}(h). \quad (9)$$

The algorithms for calculating the coefficients in (8) and (9) are different. Coefficients  $\tilde{b}'_i$  ( $i = \overline{1, 5}$ ) in

the right-hand side of equation (8) are determined as mean values of the coefficients in functions  $f_j^{(min)}(t)$ ,  $j = \overline{1, n}$ :  $\tilde{b}'_i = \frac{1}{n} \sum_{j=1}^n b'_{ij}$ ,  $i = \overline{1, 5}$ . The first term in the right-hand side of (9) can be found by using the previously mentioned gradient projection method, applying it to the multi-year average of Greenland annual seasonal temperature variability.

Suppose that the nature of changes of the mean SAT maxima and minima throughout a year is the same for the entire territory of Greenland and is linearly dependent only on the absolute height of the AWS where SAT was measured. From equations (8) and (9), we shall determine functions  $\phi^{(min)}(h)$  and  $\phi^{(max)}(h)$ , establishing dependence of the mean SAT maxima and minima on the absolute elevation:

$$\phi_j^{(min)} = f_j^{(min)}(t) - \sum_{i=1}^5 \tilde{b}'_i t^i,$$

$$\phi_j^{(max)} = f_j^{(max)}(t) - \sum_{i=1}^4 \tilde{b}''_{2i-1} \sin\left(\frac{t\pi}{365} - \tilde{b}''_{2i}\right), \quad j = \overline{1, n}.$$

The values  $\phi_j^{(min)}$  and  $\phi_j^{(max)}$  are satisfactorily approximated by linear functions with determination coefficients  $R^2 = 0.6-0.7$  (Fig. 3).

Accordingly, the final form of the functions approximating the fields of the mean SAT maxima and minima will look as follows:

$$f_{gen}^{(min)}(t, h) = \sum_{i=1}^5 \tilde{b}'_i t^i + \tilde{b}'_6 + \tilde{b}'_7 h,$$

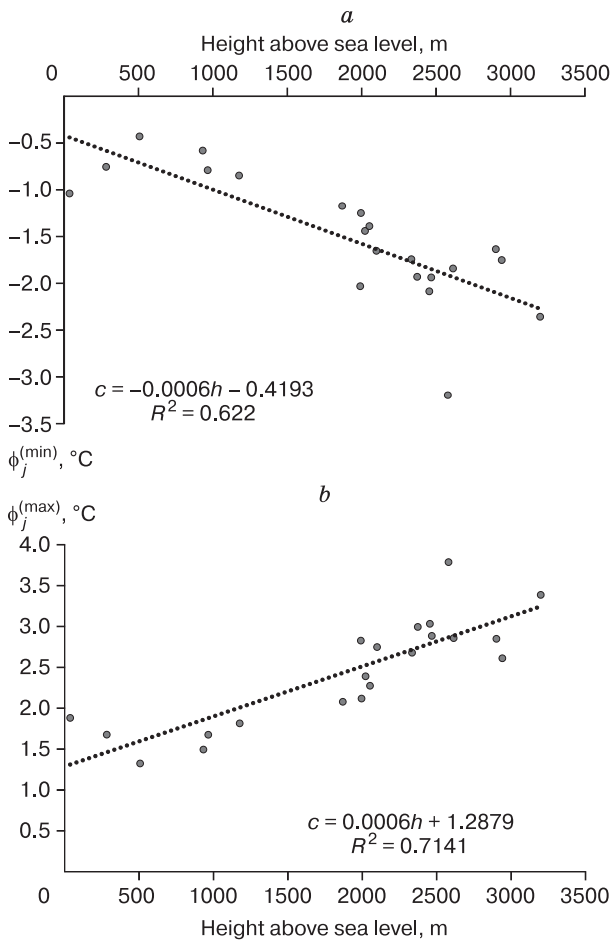
$$f_{gen}^{(max)}(t, h) = \sum_{i=1}^4 \tilde{b}''_{2i-1} \sin\left(\frac{t\pi}{365} - \tilde{b}''_{2i}\right) + \tilde{b}'_9 + \tilde{b}'_{10} h.$$

Considering the previously obtained  $\phi_j^{(min)}$ , we find:  $\tilde{b}'_1 = 0.21046$ ,  $\tilde{b}'_2 = -0.00086$ ,  $\tilde{b}'_3 = 6.95 \cdot 10^{-6}$ ,  $\tilde{b}'_4 = -2.1 \cdot 10^{-8}$ ,  $\tilde{b}'_5 = -2.16 \cdot 10^{-11}$ ,  $\tilde{b}'_6 = -0.41933$ ,  $\tilde{b}'_7 = -0.00058$ .

For the function  $f_{gen}^{(max)}(t, h)$  we obtain the following coefficients:  $\tilde{b}''_1 = 0$ ,  $\tilde{b}''_2 = 0.133134$ ,  $\tilde{b}''_3 = 0.602547$ ,  $\tilde{b}''_4 = 0$ ,  $\tilde{b}''_5 = 0.869348$ ,  $\tilde{b}''_6 = 3.026055$ ,  $\tilde{b}''_7 = 0.87741$ ,  $\tilde{b}''_8 = 1.719926$ ,  $\tilde{b}''_9 = 1.287887$ ,  $\tilde{b}''_{10} = 0.000612$ .

Shown in Fig. 4 is an example of the calculations of approximations of the mean SAT maxima and minima at AWS  $j = 7$ .

In Fig. 4, the symbol 3 indicates overshoots in the original time series of the mean SAT maxima and minima. These observations were excluded from the original series of data when selecting the approximating function  $f_{gen}^{(max)}(t, h)$ . To determine the over-



**Fig. 3. Distribution of  $\phi_j^{(min)}$  (a) and  $\phi_j^{(max)}$  (b) depending on the elevation above sea level of the  $j$ -th automated meteorological station ( $j = \overline{1, n}$ ).**

shoots, the sigma rule was used: for each month of the year, the confidential interval was calculated for the observation data of the given month (line 4 in Fig. 4). Thus, all observations  $c_{\lambda,j}^{(\max)}$ , for which the function  $f_{gen}^{(\max)}(t,h)$  was obtained, satisfied the following conditions:

$$\left| c_{\lambda,j}^{(\max)} - \overline{c_{k,j}^{(\max)}} \right| < 2.5\sigma\left(c_{k,j}^{(\max)}\right),$$

$$c_{\lambda,j}^{(\max)} > 0, \quad j = \overline{1,n}, \quad k = \overline{1,12}.$$

Here  $\overline{c_{k,j}^{(\max)}}$  is the mean value of the time series  $c_{\lambda,j}^{(\max)}$  in the  $k$ -th month;  $\sigma\left(c_{k,j}^{(\max)}\right)$  is the mean square error deviation of the time series  $c_{\lambda,j}^{(\max)}$ , calculated for month  $k$ . Coefficient 2.5 in front of the mean square error deviation was selected empirically in order to exclude obvious overshoots in the time series of the observations related to the upper limits of SAT anomalies. This limit corresponds to the following probability:

$$P\left[-2.5\sigma\left(c_{k,j}^{(\max)}\right) < c_{\lambda,j}^{(\max)} - \overline{c_{k,j}^{(\max)}} < 2.5\sigma\left(c_{k,j}^{(\max)}\right)\right] = 2 \cdot \Phi(2.5) = 0.988,$$

Table 4. Determination factors ( $R^2$ ) and factors of the mean squares error of approximation ( $\sigma_e$ ) for functions  $f_{gen}^{(\min)}(t,h)$ ,  $f_{gen}^{(\max)}(t,h)$ ,  $f_{day}(t,h,t_1)$

$j$	$f_{gen}^{(\min)}(t,h)$		$f_{gen}^{(\max)}(t,h)$		$f_{day}(t,h,t_1)$	
	$R^2$	$\sigma_e$	$R^2$	$\sigma_e$	$R^2$	$\sigma_e$
1	0.33	0.60	0.59	0.58	0.47	0.88
2	0.49	0.68	0.48	0.76	0.53	1.12
3	0.49	0.79	0.58	0.77	0.50	1.31
4	0.01	0.87	0.16	0.85	0.06	1.33
5	0.37	0.68	0.33	0.73	0.29	1.20
6	0.53	1.10	0.59	1.04	0.66	1.40
7	0.55	0.71	0.59	0.73	0.56	1.06
8	0.64	0.70	0.67	0.74	0.69	1.06
9	0.00	0.62	0.28	0.60	0.31	0.80
10	0.58	0.72	0.63	0.71	0.66	1.14
11	0.37	0.73	0.59	0.73	0.67	1.06
12	0.58	0.82	0.63	0.86	0.66	1.17
13	0.10	1.15	0.17	1.08	0.39	1.51
14	0.22	0.92	0.14	1.01	0.00	1.79
15	0.59	0.79	0.59	0.83	0.69	1.14
16	0.00	2.00	0.00	1.75	0.44	2.02
17	0.00	0.81	0.00	0.84	0.00	0.92
18	0.00	0.85	0.00	0.90	0.12	0.96
19	0.00	1.26	0.00	1.23	0.00	1.19
20	0.00	1.10	0.00	1.16	0.00	1.28
21	0.32	1.05	0.35	1.17	0.45	1.55

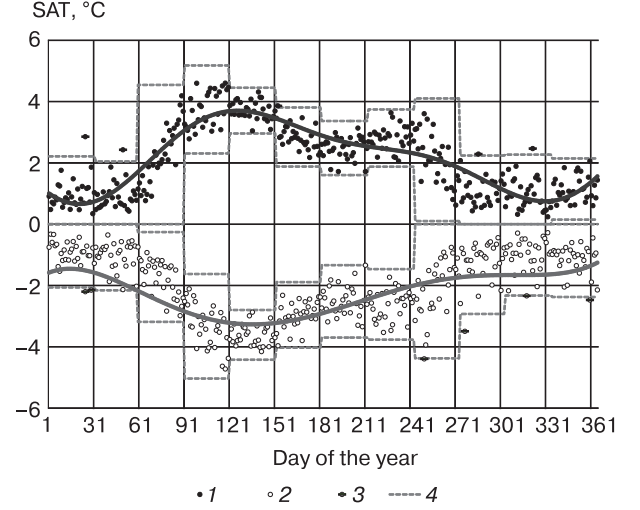


Fig. 4. Approximation of mean maxima (1) and mean minima (2) of the surface air temperature at Tunu-station ( $j = 7$ ).

3 – overshoots in the original time series of mean maxima and mean minima of SAT; 4 – confidential limits for the mean maxima and mean minima of SAT.

where  $\Phi(2.5)$  is the value of the Laplace integral function at point 2.5. In other words,

$$\Phi(2.5) = \frac{1}{\sqrt{2\pi}} \int_0^{2.5} \exp\left(-\frac{z^2}{2}\right) dz \approx 0.4938.$$

The parameters  $\overline{c_{k,j}^{(\max)}}$  and  $\sigma(c_{k,j}^{(\max)})$  were calculated as follows:

$$\overline{c_{k,j}^{(\max)}} = \frac{1}{t_{k,f} - t_{k,s} + 1} \sum_{t_k=t_{k,s}}^{t_{k,f}} c_{t_k,j}^{(\max)},$$

$$\sigma(c_{k,j}^{(\max)}) = \sqrt{\frac{1}{t_{k,f} - t_{k,s} + 1} \sum_{t_k=t_{k,s}}^{t_{k,f}} \left( c_{t_k,j}^{(\max)} - \overline{c_{k,j}^{(\max)}} \right)^2},$$

$$j = \overline{1, n}, \quad k = \overline{1, 12}.$$

Here  $t_{k,s}$  is the number of the first day of month  $k$ ;  $t_{k,f}$  is the number of the last day of month  $k$ . A similar procedure was carried out for all the initial time series.

In a similar way, overshoots are determined for the series  $c_{\lambda,j}^{(\min)} < 0$ ,  $\lambda = \overline{1, 365}$ ,  $j = \overline{1, n}$ .

Table 4 shows determination coefficients of functions  $f_{gen}^{(\min)}(t, h)$  and  $f_{gen}^{(\max)}(t, h)$ . It is clear that at certain weather stations the accuracy of approximation is either zero or close to zero. It is apparently explained by significant gaps in the initial records or by their shortness.

### 3. Parametrization of the daily variability of SAT

The daily variability of SAT is usually approximated by a harmonic function, basing on assumptions of the amplitude (or maximum and minimum). It is to be emphasized that here we mean deviations from the from multi-year average of the value of a particular day of the year. The equation approximating the daily variability of SAT will look as follows:

$$f_{day}(t, h, t_1) = B_{t,h} - A_{t,h} \sin\left(\frac{\pi t_1}{12}\right),$$

where  $t$  is the day of the year analyzed;  $t = \overline{1, 365}$ ;  $h$  is the elevation above sea level;  $t_1$  is the hour of the day  $t$ ,  $t_1 = \overline{1, 24}$ .

In the segment  $t_1 = \overline{1, 24}$  this periodic function will have one point of minimum and one point of maximum. It is evident that these points will correspond to the parametrization expressions for the mean SAT maxima and minima found in Section 2. Therefore, we can write

$$B_{t,h} = \frac{\left[ f_{gen}^{(\min)}(t, h) + f_{gen}^{(\max)}(t, h) \right]}{2},$$

$$A_{t,h} = \frac{\left[ f_{gen}^{(\max)}(t, h) - f_{gen}^{(\min)}(t, h) \right]}{2}.$$

Thus, the function approximating the daily variability of SAT will look as follows:

$$f_{day}(t, h, t_1) = \frac{\left[ f_{gen}^{(\min)}(t, h) + f_{gen}^{(\max)}(t, h) \right]}{2} - \frac{\left[ f_{gen}^{(\max)}(t, h) - f_{gen}^{(\min)}(t, h) \right]}{2} \sin\left(\frac{\pi t_1}{12}\right). \quad (10)$$

We can judge about the quality of approximation by function (10) from determination coefficients and from the mean square error calculated for each AWS (Table 4). The field of SAT anomalies is satisfactorily approximated in the area of most of the stations. An exception are those AWS for which the quality of approximation of the mean SAT maxima and minima according to the determination coefficient is zero or extremely low. The latter, in addition to the character of the SAT trends, is explained by either comparatively short duration of the time series (Table 1) or by the gaps in the time series, caused by prolonged absence of observations over SAT at particular stations.

### 4. Approximation of the SAT standard deviation

In the model calculations of the values of ablation using the temperature index method, the prescribed the value of the daily standard SAT deviation [Zweck and Huybrechts, 2005]. As a rule, the standard SAT deviation, which is, generally speaking, a tunable parameter of the model, is set uniform for the entire ice sheet. Let us check how reasonable is the assumption about its dependence on the absolute elevation. Let us determine the standard SAT deviation  $SD_t^{(j)}$  at a weather station  $j$  on the day of the year  $t$ :

$$SD_t^{(j)} = \sqrt{\frac{1}{24} \sum_{t_1=1}^{24} (x_{t,t_1,j} - c_{t,j})^2}.$$

Let us calculate the values of  $SD_k$  averaged over months of the year  $k$ :

$$SD_k = d_{1,k} + d_{2,k} h, \quad k = \overline{1, 12}.$$

Here  $d_{1,k}$ ,  $d_{2,k}$  are coefficients of the linear dependence of the parameter of the SAT standard deviation on the elevation at which the measurements were made. In the general case, they are interpreted as follows:  $d_{1,k}$  is the value of the SAT standard deviation at the sea level in month  $k$ ;  $d_{2,k}$  – the indicator of the influence of the change of elevation by a unit change of the SAT standard deviation. To find these coefficients for each month  $k$ , we shall use the least squares method, solving the optimization problem

$$F(c_1, c_2) = \sum_{i=1}^n \left( SD_k^{(i)} - d_{1,k} - d_{2,k} h^{(i)} \right)^2 \rightarrow \min, \quad k = \overline{1, 12}. \quad (11)$$

As previously, the quality of approximating real data with equation (11) was evaluated by the determination coefficient  $R^2$ . We found that acceptable accuracy of approximation ( $R^2$ ) may be obtained only for three summer months:

$$\text{June} \quad SD_k = 1.1262 + 0.0007h, \quad R^2 = 0.6035;$$

$$\text{July} \quad SD_k = 0.5151 + 0.0008h, \quad R^2 = 0.6357;$$

$$\text{August} \quad SD_k = 1.1714 + 0.0008h, \quad R^2 = 0.5456.$$

This conclusion indicates that for the months most important for ablation, dependence of standard deviation on elevation should be taken into consideration, in accordance with the parametrization equations mentioned above.

Further analysis showed that  $SD_k$  is practically independent on the latitude and longitude of AWS ( $R^2 \ll 0.5$ ).

### 5. Parametrization of the daily SAT amplitudes

SAT amplitudes  $z_{\lambda,j}$  on day  $\lambda$  in the area of station  $j$  are understood as the difference between maxima and minima of temperatures on day  $\lambda$ :

$$z_{\lambda,j} = \max_t x_{j,\lambda,t} - \min_t x_{j,\lambda,t}. \quad (12)$$

Note that the SAT amplitudes may be approximated by the difference of the mean SAT maxima and minima (Section 2), as the approximated value of

$$c_{\lambda,j}^{(\max)} - c_{\lambda,j}^{(\min)} = \max_t x_{j,\lambda,t} - a_{j,\lambda} - \left( \max_t x_{j,\lambda,t} - a_{j,\lambda} \right),$$

where  $a_{j,\lambda}$  is the mean SAT on day  $\lambda$  in the area of station  $j$ , agrees with definition (12). Yet, to simplify practical calculations, we shall adjust the approximating function directly to the data series  $z_{\lambda,j}$ .

To select the type of the approximating function, consider distribution of SAT amplitudes by days during a year, averaged for all the stations (Fig. 5).

The character of the distribution presupposes the function describing these changes to contain four extreme points. Preliminary estimates have shown that approximation with a harmonic function is more

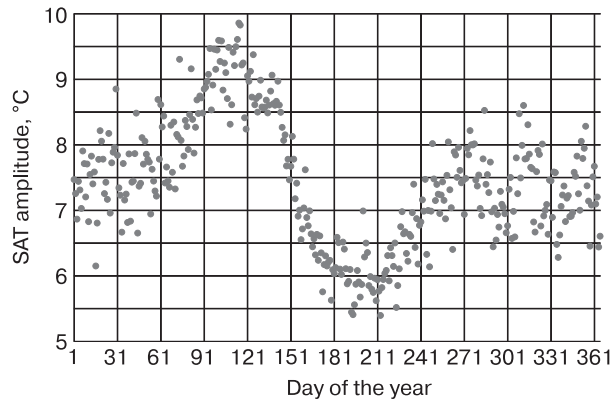


Fig. 5. Mean daily SAT amplitude for Greenland.

Table 5. Coefficients  $A_{j_h}, B_{j_h}, C_{j_h}$  of function (13) for the  $j$ -th automatic weather station

$j_h$	$A$	$B$	$C$
1	-1.22	1.05	5.97
2	0.57	3.84	7.99
3	0.81	4.18	8.19
4	0.78	3.42	5.05
5	0.58	3.94	6.95
6	-1.57	0.96	10.9
7	0.48	4.44	7.93
8	-1.08	13.4	8.87
9	0.99	16.56	5.06
10	-0.99	13.63	8.99
11	-1.06	7.60	8.69
12	-1.14	1.55	9.04
13	0.63	10.72	8.03
14	0.86	16.96	7.74
15	0.64	23.00	9.07
16	0.24	15.82	8.91
17	0.77	16.68	4.89
18	0.69	10.38	4.84
19	-0.97	13.83	5.70
20	-0.99	25.73	4.56
21	1.25	41.99	8.67

preferable than that with the polynomial function. Let us set the parametric function  $f(t,h)$  as

$$f(t,h) = A_h \sin(\omega t + B_h) + C_h; \quad (13)$$

$$A_h = d_{A,0} + d_{A,1}h; \quad (14)$$

$$B_h = d_{B,0} + d_{B,1}h; \quad (15)$$

$$C_h = d_{C,0} + d_{C,1}h. \quad (16)$$

It follows from (13) that the amplitude of periodic function  $f(t,h)$  depends on coefficient  $A_h$ , the

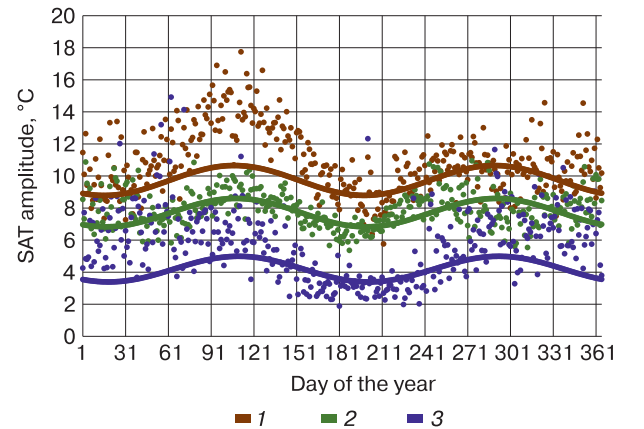
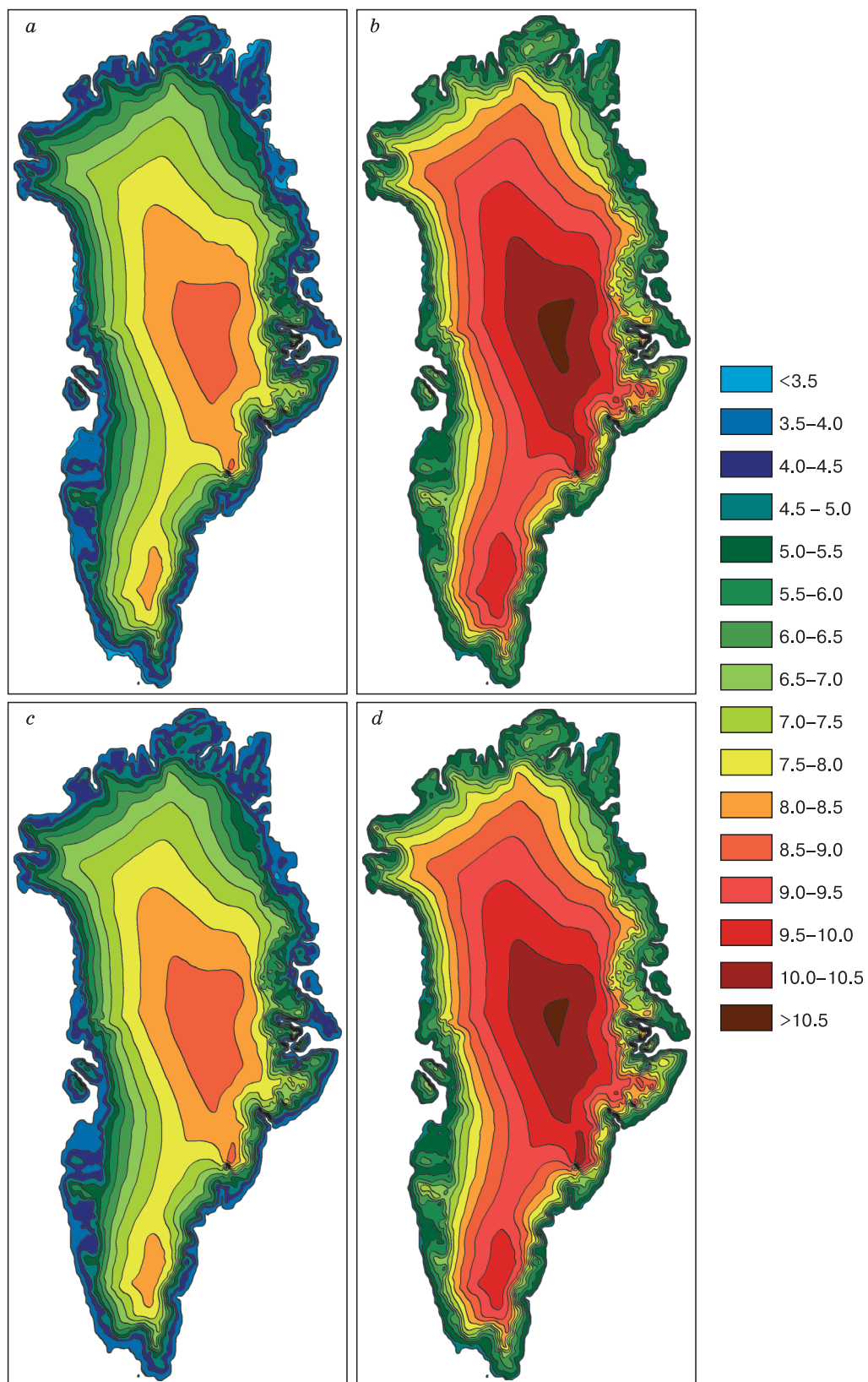


Fig. 6. Approximating curves of the SAT amplitude depending on the elevation.

1 – in the area of Summit station (3199 m asl); 2 – in the area of Tunu-N station (2052 m asl); 3 – in the area of Petermann Gl station (37 m asl).



**Fig. 7. Mean monthly day amplitudes of temperatures (°C).**

*a* – January; *b* – April; *c* – July; *d* – October.

shift of function  $f(t, h)$  on the X-axis depends on coefficient  $B_h$ , and the shift of function  $f(t, h)$  on the Y-axis depends on coefficient  $C_h$ .

Each of the parameters,  $A_h$ ,  $B_h$  and  $C_h$ , according to assumptions (14)–(16), linearly depends on the elevation. Coefficients  $d_{i,0}$ ,  $d_{i,1}$ ,  $i = A, B, C$  for each parameter  $i_h$  are interpreted as follows:  $d_{i,0}$  – the value of parameter  $i_h$  in equation (13) for  $h = 0$  (at sea level);  $d_{i,1}$  – the indicator of how the change in the height by 1 m influences the change in the respective parameters  $i_h$ .

To approximate parameters  $A_h$ ,  $B_h$  и  $C_h$  for the arbitrary elevation  $h$ , coefficients  $d_{i,0}$ ,  $d_{i,1}$ ,  $i = A, B, C$  must be found. They can be calculated by the gradient method from the assumption that for all AWS the sums of the squares of deviations of the observed SAT amplitudes from those calculated with parametric function  $f(t, h)$  will be minimal:

$$\sum_{t=1}^{365} [z_{t,j} - f(t, h_j)]^2 \rightarrow \min, \quad j = \overline{1, n}, \quad (17)$$

$$A_h > 0, \quad 0 \leq B_h < 2\pi.$$

Here  $f(t, h_j)$  is the value of the daily SAT amplitude on a day of year  $t$  at station  $j$  (located at elevation  $h_j$  above sea level) approximated with periodic function (13);  $n$  is the number of stations.

It is evident that in order to solve problem (17), it is necessary to know the general period  $w$  of functions (13). Since the annual temperature amplitudes contain four extreme points (two maxima and two minima), the period of function  $f(t, h)$  will be equal to half-year, i.e., two periods of function  $f(t, h)$  take place during a year. Hence,  $w = \frac{4\pi}{365}$ .

Applying the gradient method, we obtain coefficients of the parametric function (13)  $A_{j_h}$ ,  $B_{j_h}$ ,  $C_{j_h}$  for stations  $j = \overline{1, n}$  (Table 5). Using the least squares method, we can calculate the linear dependence between coefficients  $A_{j_h}$ ,  $B_{j_h}$ ,  $C_{j_h}$  and  $h$ :  $A_h = 0.798\,055 + 0.000\,042h$ ,  $B_h = 4.053\,022 + 0.000\,033h$ ,  $C_h = 4.121\,33 + 0.001\,749h$ . As an example of calculations in accordance with (13)–(16), shown in Fig. 6 are the approximating curves for SAT amplitudes for *Summit*, *Tunu-N* and *Petermann Gl* AWS.

The results of the calculations averaged for January, April, July, and October are shown in Fig. 7. The maximum amplitudes correspond to April and October, and the minimum amplitudes correspond to January and July. This conclusion is important for model estimations of ablation, as it demonstrates the necessity of considering variations in the amplitudes of daily temperatures during a year. The use of the constant value of the amplitude may lead to overestimation of the calculated amount of melted water.

## CONCLUSIONS

The paper proposes methods of approximating such characteristics of the surface air temperature in Greenland as standard deviation, daily amplitudes, daily minima and maxima, and daily temperature variability. The calculations were based on hourly meteorological observations conducted at automatic weather stations located over the entire range of altitudes. Analysis of the original information resulted in the following conclusions:

- the mean maxima, mean minima, and amplitudes of the daily SAT are clearly seasonal: they are relatively greater in spring and in autumn and are relatively smaller in winter and in summer;
- the mean maxima are best approximated by the harmonic function, and the mean minima are better approximated by the polynomial function;
- determination coefficients characterizing the quality of approximation exceed 0.5 only for half of the stations; the latter can be explained either by relative shortness of the original time series or by the gaps in the data or by variability of SAT itself;
- all the three indicated characteristics have clear linear dependence on the absolute elevation;
- at the same time, approximation of the mean square deviation of SAT depending on the absolute elevation is significant only for summer months;
- approximation of the daily variability of SAT demonstrates that in some cases there are deviations from the harmonic character of SAT variations during a day.

The results obtained may be useful in developing mathematical models of the surface mass balance, as they provide significant details of the schematic characteristics of SAT variations.

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